

1. Assume the temperature $u(x, y) = \ln(x^2 + y^2)$ is a function of position (x, y) for the region $\{(x, y) | x^2 + y^2 \geq 1\}$. A particle moves in the region. The position of the particle at time t is $(x^*(t), y^*(t))$, where $x^*(t) = t \cos(t)$ and $y^*(t) = t \sin(t)$.
- (a) Find the unit tangent vector $T(t)$ and the unit normal vector $N(t)$ for the path of the particle. 4%
- (b) Find the velocity and the acceleration of the particle. 4%
- (c) Find the temperature change rate at the particle, i.e. $\frac{d}{dt}u(x^*, y^*)$. 4%
- (d) Find the gradient of the temperature at (x, y) , i.e. $\nabla u(x, y)$. 4%
- (e) Show that the divergence of ∇u is zero, i.e. $\nabla \cdot \nabla u = 0$. 4%
- (f) The heat flux is $D\nabla u(x, y)$ where is D a positive constant. Find the total heat flow which diffuse from outside into the region. 5%
2. Let $\sum_{i=1}^{\infty} a_i$ be a real series.
- (a) State the definition of " $\sum_{i=1}^{\infty} a_i = L$ " (the series converges to L). 5%
- (b) Prove that "If $\sum_{i=1}^{\infty} a_i$ converges then $\lim_{i \rightarrow \infty} a_i = 0$ ". 5%
- (c) Prove that "If $\sum_{i=1}^{\infty} |a_i|$ converges then $\sum_{i=1}^{\infty} a_i$ converges". 5%
3. Let $f(x) = \int_1^x \exp(t^3) dt$
- (a) Find $f(1)$. 5%
- (b) Find $f'(1)$. 5%
- (c) Find $\int_0^1 x f(x) dx$. 5%
4. Let $f(x)$ be twice differentiable on an open interval containing a and b , and $|\frac{\partial^2}{\partial x^2} f(x)| < M$ for $x \in [a, b]$. Let $g(x)$ be the linear interpolation for $f(x)$; i.e. $g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$. Prove that $|f(x) - g(x)| < \frac{|b - a|^2}{4} M$ for $x \in [a, b]$. 15%

5. Evaluate the following integrals

(a) $\int_0^{\pi} \sin^3 x dx.$ 5%

(b) $\int_{-\infty}^0 e^{2x} \cos x dx.$ (improper integral) 5%

(c) $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \frac{y}{\sqrt{x^2+y^2}} dy dx.$ (double integral) 5%

6. Evaluate the following limits

(a) $\lim_{x \rightarrow \infty} x \sin \frac{1}{x}.$ 5%

(b) $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{(1 + \frac{1}{n})^k}.$ 5%

(c) $\lim_{x \rightarrow 0} \frac{\sin^{-1}(a+x) - \sin^{-1}(a-x)}{x}.$ 5%