1. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$T\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2a_1 - a_2 \\ 3a_1 + 2a_2 \\ a_1 - a_2 \end{pmatrix}.$$

   (a) Let $\beta$ and $\gamma$ be the standard ordered basis for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, find the representation matrix $[T]_\beta^\gamma$. (3%)

   (b) Let $\beta' = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ and $\gamma' = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$, compute $[T]_{\beta'}^{\gamma'}$. (6%)

   (c) Find a $2 \times 2$ matrix $P$ and a $3 \times 3$ matrix $Q$ such that $[T]_\beta^\gamma = Q[T]_{\beta'}^{\gamma'}P$. (6%)

2. Solve the system of linear equations:

$$\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 1 \\ x_1 - x_2 + 2x_3 = 2 \\ 2x_1 + 3x_3 + x_4 = 3. \end{cases}$$ (10%)

3. Let $V$ and $W$ be finite-dimensional vector spaces and $T : V \to W$ be linear. Suppose $V_0$ is a subspace of $V$.

   (a) Prove that $T(V_0)$ is a subspace of $W$. (5%)

   (b) Find the relation of $\dim(V_0)$ and $\dim(T(V_0))$, and prove your conclusion. (12%)

4. Suppose $A$ and $B$ are two $n \times n$ matrices and $A$ is similar to $B$.

   (a) Show that $A$ and $B$ have the same characteristic polynomial. (7%)

   (b) Show that trace $A$ = trace $B$. (7%)

   (c) Let $(-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ be the characteristic polynomial of $A$, show that $A$ is invertible if and only if $a_0 \neq 0$. (6%)

5. Let $A = \begin{pmatrix} 1/2 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 3/4 & 1/4 \\ 0 & 1/4 & 3/4 \end{pmatrix}$.

   (a) Find the Jordan cononical form of $A$ (say $J$). (6%)

   (b) Find an invertible matrix $S$ such that

$$S^{-1}AS = J.$$ (8%)

   (c) Evaluate $\lim_{n\to\infty} A^n$. (8%)

6. Let $W = \text{span}\{(1,1,1,1),(1,1,-1,0)\}$ and $P$ be the orthogonal projection from $\mathbb{R}^4$ onto $W$.

   (a) Compute $P(x_1, x_2, x_3, x_4)$. (10%)

   (b) Find the orthogonal projection of $u = (1,2,3,4)$ on $W^\perp$. (6%)