1. Let $A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$ be a 8×8 matrix.

   (i) Find the characteristic polynomial and minimal polynomial of matrix A. (7%)

   (ii) Let B be a 8×8 matrix. Suppose that A and B have the same characteristic polynomial and minimal polynomial. Find all possible Jordan normal forms of B. (3%)

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ with $T(x, y, z) = (y, z, x)$. Find all subspaces W with $T(W) \subseteq W$. (10%)

3. Let V be a vector space over a field k, and let $T : V \to V$ be a linear transformation. Let $v_1, \ldots, v_m$ be eigenvectors of T, with eigenvalues $\lambda_1, \ldots, \lambda_m$ respectively. Assume that $\lambda_i \neq \lambda_j$ if $i \neq j$. Show that $v_1, \ldots, v_m$ are linearly independent. (10%)

4. (i) Let V be a vector space over $\mathbb{R}$ with an inner product $\langle \ , \ \rangle$. Let W be a finite dimensional vector subspace with orthonormal basis $\{w_1, \ldots, w_k\}$. Let $v \in V$ and $d = \inf\{\|v - w\| \in \mathbb{R} \mid w \in W\}$. Show that $d = \|v - \langle v, w_1\rangle w_1 - \cdots - \langle v, w_k\rangle w_k\|$. ([illegible]%)

   (ii) Let $I : \mathbb{R}^2 \to \mathbb{R}$ with

$$I(a, b) = \int_0^{\frac{\pi}{2}} \left|\sin x - (ax + b)\right|^2 dx.$$

   Find $(\alpha, \beta) \in \mathbb{R}^2$ such that $I(\alpha, \beta) \leq I(a, b)$ for all $(a, b) \in \mathbb{R}^2$. (8%)

5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n$ diverges. Show that

   (i) $\sum_{n=1}^{\infty} \frac{a_n}{1 + a_n}$ diverges; (5%)

   (ii) $\sum_{n=1}^{\infty} \frac{a_n}{1 + n^2 a_n}$ converges; (5%)

   (iii) $\sum_{n=1}^{\infty} \frac{a_n}{1 + n a_n}$ sometimes converges and sometimes diverges. (10%)

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at some point p in $\mathbb{R}$. Put $f^+ : \mathbb{R} \to [0, \infty)$: $f^+(x) = \max\{f(x), 0\}$. Show that the function $g : \mathbb{R} \to \mathbb{R} : g(x) = \left(f^+(x)\right)^2$ is differentiable at p with $g'(p) = 2f^+(p)f'(p)$. (15%)

7. Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Is the function $g : [a, b] \to \mathbb{R} : g(x) = \sup_{a \leq t \leq x} f(t)$ continuous? (7%)

8. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and satisfies

$$\int_0^1 f(x)x^n \, dx = 0, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

   Find the range of f. (8%)