國立成功大學 103 學年度「博士班」研究生招生入學考試

【高等微積分】

Advanced Calculus

1. (25) State and prove the Heine-Borel theorem.
2. (25) State and prove the inverse function theorem for $\mathbb{R}^n$.
3. (25) State and prove the closed graph theorem for a real-valued function.
4. (25) Evaluate the line integral

$$\int_\gamma F,$$

where

$$\gamma : [0,\pi] \to \mathbb{R}^2, \quad \gamma(t) = (\sin(\pi e^{\sin t}), \cos^5 t)$$

and

$$F(x,y) = (e^x(\sin(x+y)+\cos(x+y))+1, e^x\cos(x+y)).$$

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【線性代數】

Linear Algebra

Notice: Justify your answers to get the full credit.

1. (10 points) Let

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & -1 & 0 \\ -2 & 1 & 4 & -1 & 3 \\ 3 & -1 & -5 & 1 & -6 \end{pmatrix}.$$

Consider the set $\Gamma = \{B \in M_{5\times 5}(\mathbb{R}) \mid AB = 0\}$. Find the number

$$\min_{B\in\Gamma} \{\dim_{\mathbb{R}}\langle v \in M_{5\times 1}(\mathbb{R}) \mid Bv = 0\rangle\}.$$

2. (10 points) Let $y_1, y_2, \ldots, y_n$ be linear independent functions in C^∞, the vector space of all C^∞-functions. Consider the linear transformation $T : C^\infty \to C^\infty$ defined by

$$[T(y)](t) = \det \begin{pmatrix} y(t) & y_1(t) & y_2(t) & \cdots & y_n(t) \\ y'(t) & y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \vdots & & \vdots \\ y^{(n)}(t) & y_1^{(n)}(t) & y_2^{(n)}(t) & \cdots & y_n^{(n)}(t) \end{pmatrix}.$$

Find the null space of T.

3. Let

$$A = \begin{pmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{pmatrix}.$$

(a) (10 points) Find an invertible matrix Q such that $Q^{-1}AQ$ is the Jordan canonical form of A.

(b) (10 points) Does the limit $\lim_{m\to\infty} A^m$ exists?

(c) (10 points) Find a diagonalizable matrix B in $M_{3\times 3}(\mathbb{R})$ such that $N = A - B$ is nilpotent (*i.e.* $N^m = 0$ for some non-negative integer m) and NB=BN.

4. Let

$$A = \begin{pmatrix} 0 & 2 & 0 & -6 & 2 \\ 1 & -2 & 0 & 0 & 2 \\ 1 & 0 & 1 & -3 & 2 \\ 1 & -2 & 1 & -1 & 2 \\ 1 & -4 & 3 & -3 & 4 \end{pmatrix}.$$

(a) (10 points) Find the minimal polynomial of A.

(b) (10 points) Find the rational canonical form of A.

5. (a) (15 points) Let T be a diagonalizable linear operator on a finite-dimensional vector space. Show that the restriction of T to any nontrivial T-invariant subspace is also diagonalizable.

(b) (15 points) Show that two commuting real symmetric matrices are simultaneously diagonalizable. (*i.e.* If A and B are two real symmetric matrices satisfying $AB = BA$, then there exists an invertible matrix Q such that $Q^{-1}AQ$ and $Q^{-1}BQ$ are both diagonal matrices.)

【實變數函數論】

Real Analysis

1. (a) State the Fatou's lemma. (20 points)

(b) Given an example to show that the inequality in Fatou's lemma may be strict. (20 points)

2. Give an example of a sequence of Lebesgue integrable functions f_n converging everywhere to a Lebesgue integrable function f, such that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)dx < \int_{-\infty}^{\infty} f(x)dx. \quad \text{(20 points)}$$

3. Give an example of a sequence of functions f_n on $[a,b]$ such that each f_n is Riemann integrable, $|f_n| \leq 1$ for all n, $f_n \to f$ everywhere, but f is not Riemann integrable. (20 points)

4. Show that $\int_1^\infty e^{-t}\ln t\,dt = \lim_{n\to\infty}\int_1^n [1-(t/n)]^n \ln t\,dt$. (20 points)

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【代數】

Algebra Exam Date: Thursday 01/05/2014

Do all the following problems. Be sure to show all work and explain your reasoning as clearly as possible.

1. (a) (10%) Prove that for $n \geq 5$, the only normal subgroups of S_n are 1, A_n and S_n.
   (b) (10%) Prove that if p is a prime and P is a non-abelian group of order p^3 then $|Z(P)| = p$ and $P/Z(P) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$.
   (c) (10%) Prove that if $|G| = 1365$ then G is not simple.

2. (a) (10%) Let p and q are primes with $p < q$. Show that every group of order p^2q is solvable.
   (b) (10%) Show that a finite group is nilpotent if and only if every maximal subgroup is normal.

3. Let ζ_n be a primitive nth root of unity. Define the nth *cyclotomic polynomial* $\Phi_n(x)$ as follows:

$$\Phi_n(x) = \prod_{\substack{1 \leq a < n \\ \gcd(a,n)=1}} (x - \zeta_n^a)$$

   (a) (20%) Show that $\Phi_n(x)$ is an irreducible monic polynomial in $\mathbb{Z}[x]$ of degree $\varphi(n)$, where φ denotes Euler's phi-function.
   (b) (5%) Conclude using (a) that

$$[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n).$$

4. (25%) Determine the Galois group of $(x^2 - 2)(x^2 - 3)(x^2 - 5)$. Determine all the subfields of the splitting field of this polynomial.