On the Dirac-Klein-Gordon Equations in one Space Dimension

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Abstract. We establish local and global existence results for Dirac-Klein-Gordon equations in one space dimension, employing a null form estimate and a fixed point argument.

0. Introduction and Main Results.

In the present work, we like to study the Cauchy problem for the Dirac-Klein-Gordon equations. The unknown quantities are a spinor field $\psi : \mathbb{R} \times \mathbb{R}^1 \mapsto \mathbb{C}^4$ and a scalar field $\phi : \mathbb{R} \times \mathbb{R}^1 \mapsto \mathbb{R}$. The evolution equations for the fields are given below,

$$\mathcal{D}\psi = \phi\psi; \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^1 \tag{0.1a}$$

$$\Box\phi = \overline{\psi}\psi; \tag{0.1b}$$

$$\psi(0, x) := \psi_0(x), \quad \phi(0, x) = \phi_0(x), \quad \phi_t(0, x) = \phi_1(x), \tag{0.1c}$$

where $\mathcal{D}$ is the Dirac operator, $\mathcal{D} := -i\gamma^\mu \partial_\mu$, $\mu = 0, 1$, and γ^μ are the Dirac matrices, the wave operator $\Box = -\partial_{tt} + \partial_{xx}$, and $\overline{\psi} = \psi^\dagger \gamma^0$, and $\dagger$ is the complex conjugate transpose.

The purpose of this work is to demonstrate the usefulness of a null form estimate, by employing the solution representations in Fourier transform of the DKG equations. We will take full advantage of the null form structure depicted in the nonlinear term $\overline{\psi}\psi$, which has been observed for possessing such structure, see [KM] and [Bo].

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For the DKG system, there are many conserved quantities which are not positive definite, such as the energy. Therefore they are not applicable to derive a priori estimates. However the known positive conserved quantity is the law of conservation of charge,

$$\int |\psi(t)|^2 \, dx \; = \text{constant} \tag{0.2}$$

which leads to the global existence result, once the local existence result is established, see [Bo] and [F2].

In '73, Chadam showed that the Cauchy problem for the DKG equations has a global unique solution for $\psi_0 \in H^1$, $\phi_0 \in H^1$, $\phi_1 \in L^2$, see [C]. In '93, Zheng proved that there exists a global weak solution to the Cauchy problem of a modified DKG equations, based on the technique of compensated compactness, with $\psi_0 \in L^2$, $\phi_0 \in H^1$, $\phi_1 \in L^2$, see [Z]. In '00, Bournaveas derived a new proof of a global existence for the DKG equations, based on a null form estimate, if $\psi_0 \in L^2$, $\phi_0 \in H^1$, $\phi_1 \in L^2$, see [B]. In '02, Fang gave a direct proof for (0.1), based on a variant null form estimate, which is more straight forward, and the result is parallel to Bournaveas', see [F2]

The outline of this paper is as follows. First we derive some solutions representations in Fourier transform. Next we prove some a priori estimates of solutions for Dirac equation and for wave equation. Then we show a local result for (0.1), employing the null form estimate together with other estimates derived previously, and a fixed point argument. Finally we show the key estimate, namely the null form estimate.

The main result in this work is as follows.

Theorem 0.1. *(Local Existence) Let* $0 < \epsilon \leq \frac{1}{4}$ *and* $0 < \delta \leq 2\epsilon$*. If the initial data of (0.1)* $\psi_0 \in H^{-\frac{1}{4}+\epsilon}$*,* $\phi_0 \in H^{\frac{1}{2}+\delta}$*,* $\phi_1 \in H^{-\frac{1}{2}+\delta}$*, then there is a unique local solution for (0.1).*

Theorem 0.2. *(Global Existence) Let* $\delta > 0$*. If the initial data of (0.1)* $\psi_0 \in L^2$*,* $\phi_0 \in H^{\frac{1}{2}+\delta}$*,* $\phi_1 \in H^{-\frac{1}{2}+\delta}$*, then there is a unique global solution for (0.1).*

Remarks.

1. The DKG equations follow from the Lagrangian

$$\int_{\mathbb{R}^{1+1}} \Big\{ |\nabla\phi|^2 - |\phi_t|^2 - \overline{\psi}\mathcal{D}\psi - \phi\overline{\psi}\psi \Big\} dxdt. \tag{0.3}$$

2. The Dirac-Klein-Gordon system must be

$$\begin{cases} \mathcal{D}\psi = \phi\psi; \\ \Box\phi + m^2\phi = \overline{\psi}\psi, \end{cases} \tag{0.4}$$

and the proof works for this system too.

3. $\widehat{\mathcal{D}}^2 = \widehat{\Box} I$, where I is the 4×4 identity matrix.

4. $\overline{\psi}\psi = \psi^\dagger\gamma^0\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$, where ψ_j are the component functions of the vector function ψ, which take values in $\mathbb{C}$.

The case $\delta = 0$ is critical in the following sense. Assuming that the initial data (ϕ_0, ϕ_1) are in $H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$ does not imply that $\phi(t, \cdot)$ is bounded. In fact, it is a BMO function. One of the motivations for proving the existence of global solution with low regularity, is based on an observation made by Grillakis, which is that the initial data of (0.1): $\psi_0 \in L^2$, $\phi_0 \in H^{\frac{1}{2}}$, $\phi_1 \in H^{-\frac{1}{2}}$, is a right space for the existence of an invariant measure, see [B] and [Ku], resulted from the DKG equations.

1. Solution Representation.

In what follows, we denote by (t, x) the time-space variables and by (τ, ξ) the dual variables with respect to the Fourier transform of a given function. We will use $\alpha = \dfrac{1}{4} - \epsilon$ throughout the paper. We will also often skip the constant in the inequalities. For convenience, we denote the multipliers by

$$\widehat{E}(\tau, \xi) = |\tau| + |\xi| + 1 \tag{1.1a}$$

$$\widehat{S}(\tau, \xi) = \Big| |\tau| - |\xi| \Big| + 1 \tag{1.1b}$$

$$\widehat{W}(\tau, \xi) = \tau^2 - |\xi|^2 \tag{1.1c}$$

$$\widehat{D}(\tau, \xi) = \gamma^0\tau + \gamma^1\xi \tag{1.1d}$$

$$\widehat{M}(\xi) = |\xi| + 1 \tag{1.1e}$$

Notice that $\widehat{W}$ and $\widehat{D}$ are the symbols of the wave and Dirac operators respectively.

Consider the Dirac equation,

$$\begin{cases} \mathcal{D}\psi = G, \quad (t,x) \in \mathbb{R}^1 \times \mathbb{R}^1, \\ \psi(0) = \psi_0. \end{cases} \tag{1.2}$$

First by taking the Fourier transform on (1.2) over the space variable and solving the resulting ODE, we can formally write down the solution as follows.

$$\widetilde{\psi}(t,\xi) = \frac{e^{it|\xi|}}{2|\xi|}\widehat{D}(|\xi|,\xi)\gamma^0\widehat{\psi}_0(\xi) + \frac{e^{-it|\xi|}}{2|\xi|}\widehat{D}(|\xi|,-\xi)\gamma^0\widehat{\psi}_0(\xi) + \int_0^t \frac{e^{i(t-s)|\xi|}}{2|\xi|}\widehat{D}(|\xi|,\xi)i\widetilde{G}(s,\xi)\,ds + \int_0^t \frac{e^{-i(t-s)|\xi|}}{2|\xi|}\widehat{D}(|\xi|,-\xi)i\widetilde{G}(s,\xi)\,ds. \tag{1.3}$$

Rewriting the inhomogeneous terms in (1.3) gives

$$\widetilde{\psi}(t,\xi) = \Big[\frac{e^{it|\xi|}}{2|\xi|}\widehat{D}(|\xi|,\xi) + \frac{e^{-it|\xi|}}{2|\xi|}\widehat{D}(|\xi|,-\xi)\Big]\gamma^0\widehat{\psi}_0(\xi) + \int \Big[\frac{e^{it\tau} - e^{it|\xi|}}{2|\xi|(\tau - |\xi|)}\widehat{D}(|\xi|,\xi) + \frac{e^{it\tau} - e^{-it|\xi|}}{2|\xi|(\tau + |\xi|)}\widehat{D}(|\xi|,-\xi)\Big]\widehat{G}(\tau,\xi)d\tau. \tag{1.4}$$

Now we split the function $\widehat{G}$ into several parts in the following manner. Consider $\widehat{a}(\tau)$ a cut-off function equals 1 if $|\tau| \le \frac{1}{2}$ and equals 0 if $|\tau| \ge 1$, and denote by $h(\tau)$ the Heaviside function. For simplicity, let us write

$$\widehat{G}_\pm(\tau,\xi) := h(\pm\tau)\widehat{a}(\tau \mp |\xi|)\widehat{G}(\tau,\xi), \tag{1.5a}$$

$$\widehat{G}_f(\tau,\xi) := \widehat{G}(\tau,\xi) - \big(\widehat{G}_+(\tau,\xi) + \widehat{G}_-(\tau,\xi)\big), \tag{1.5b}$$

$$\widehat{D}_\pm := \widehat{D}(|\xi|, \pm\xi). \tag{1.5c}$$

Notice that $\widehat{G}_\pm$ are supported in the regions $\{(\tau,\xi) : \pm\tau > 0, |\tau \mp |\xi|| \le 1\}$ respectively. Using the decomposition of the forcing term $\widehat{G} = \widehat{G}_f + \widehat{G}_+ + \widehat{G}_-$, the inhomogeneous term in (1.4) can be written as

$$\int \Big[\frac{e^{it\tau} - e^{it|\xi|}}{2|\xi|(\tau - |\xi|)}\widehat{D}(|\xi|,\xi) + \frac{e^{it\tau} - e^{-it|\xi|}}{2|\xi|(\tau + |\xi|)}\widehat{D}(|\xi|,-\xi)\Big]\widehat{G}_f(\tau,\xi)d\tau = \int e^{it\tau}\frac{\widehat{D}(\tau,\xi)}{\tau^2 - |\xi|^2}\widehat{G}_f\,d\tau - e^{it|\xi|}\frac{\widehat{D}_+}{2|\xi|}\int \frac{\widehat{G}_f}{\tau - |\xi|}d\tau - e^{-it|\xi|}\frac{\widehat{D}_-}{2|\xi|}\int \frac{\widehat{G}_f}{\tau + |\xi|}d\tau, \tag{1.6a}$$

$$\int \frac{e^{it\tau} - e^{it|\xi|}}{2|\xi|(\tau - |\xi|)} \widehat{D}_+(\widehat{G}_+ + \widehat{G}_-)d\tau$$
$$= e^{it|\xi|} \frac{\widehat{D}_+}{2|\xi|} \int \frac{e^{it(\tau-|\xi|)} - 1}{\tau - |\xi|} (\widehat{G}_+ + \widehat{a}_6(\tau)\widehat{G}_-)d\tau +$$
$$\int e^{it\tau} \frac{(1 - \widehat{a}_6(\tau))\widehat{D}_+\widehat{G}_-}{2|\xi|(\tau - |\xi|)} d\tau - e^{it|\xi|} \frac{\widehat{D}_+}{2|\xi|} \int \frac{(1 - \widehat{a}_6(\tau))\widehat{G}_-}{\tau - |\xi|} d\tau, \quad (1.6b)$$

$$\int \frac{e^{it\tau} - e^{-it|\xi|}}{2|\xi|(\tau + |\xi|)} \widehat{D}_-(\widehat{G}_+ + \widehat{G}_-)d\tau$$
$$= e^{-it|\xi|} \frac{\widehat{D}_-}{2|\xi|} \int \frac{e^{it(\tau+|\xi|)} - 1}{\tau + |\xi|} (\widehat{a}_6(\tau)\widehat{G}_+ + \widehat{G}_-)d\tau +$$
$$\int e^{it\tau} \frac{(1 - \widehat{a}_6(\tau))\widehat{D}_-\widehat{G}_+}{2|\xi|(\tau + |\xi|)} d\tau - e^{-it|\xi|} \frac{\widehat{D}_-}{2|\xi|} \int \frac{(1 - \widehat{a}_6(\tau))\widehat{G}_+}{\tau + |\xi|} d\tau, \quad (1.6c)$$

where $\widehat{a}_6(\tau) = \widehat{a}(\frac{\tau}{6})$ and $\widehat{a}$ is the cut-off function defined previously. Recall the power expansion

$$e^{it(\tau \pm |\xi|)} - 1 = \sum_{k=1}^{\infty} \frac{1}{k!} (it)^k (\tau \pm |\xi|)^k. \quad (1.7)$$

Combining (1.4)-(1.7), we can give a formula for $\widehat{\psi}$, namely

$$\widehat{\psi}(\tau, \xi) = \sum_{k=0}^{\infty} \Big(\delta_+^{(k)}(\tau, \xi)\widehat{A}_{+,k}(\xi) + \delta_-^{(k)}(\tau, \xi)\widehat{A}_{-,k}(\xi) \Big) + \widehat{K}(\tau, \xi), \quad (1.8)$$

where $\delta_\pm(\tau, \xi)$ are the delta functions supported on $\{\tau = \pm|\xi|\}$ respectively, $\delta^{(k)}$ mean derivatives of the delta function, and

$$\widehat{K}(\tau, \xi) := \frac{\widehat{D}(\tau, \xi)}{\widehat{W}(\tau, \xi)} \widehat{G}_f + \frac{(1 - \widehat{a}_6(\tau))\widehat{D}_+\widehat{G}_-}{2|\xi|(\tau - |\xi|)} + \frac{(1 - \widehat{a}_6)\widehat{D}_-\widehat{G}_+}{2|\xi|(\tau + |\xi|)}, \quad (1.9a)$$
$$\widehat{A}_{\pm,0}(\xi) := \frac{\widehat{D}_\pm}{2|\xi|} \Big[\gamma^0 \widehat{\psi}_0 - \int \frac{\widehat{G}_f + (1 - \widehat{a}_6(\lambda))\widehat{G}_\mp}{\lambda \mp |\xi|} d\lambda \Big], \quad (1.9b)$$
$$\widehat{A}_{\pm,k}(\xi) := \frac{\widehat{D}_\pm(-1)^k}{2|\xi| k!} \int (\lambda \mp |\xi|)^{k-1} \big[\widehat{G}_\pm + \widehat{a}_6(\lambda)\widehat{G}_\mp \big] d\lambda. \quad (1.9c)$$

Consider the wave equation,

$$\begin{cases} \Box\phi = F, \quad (t,x) \in \mathbb{R}^1 \times \mathbb{R}^1, \\ \phi(0) = \phi_0, \;\; \phi_t(0) = \phi_1. \end{cases} \tag{1.10}$$

Taking Fourier transform on (1.13) and solving the resulting ODE gives

$$\widetilde{\phi}(t,\xi) = \cos t|\xi|\widehat{\phi}_0(\xi) + \frac{\sin t|\xi|}{|\xi|}\widehat{\phi}_1(\xi) - \int_0^t \frac{\sin{(t-s)|\xi|}}{|\xi|}\widetilde{F}(s,\xi)ds. \tag{1.11}$$

$$\widetilde{\phi}(t,\xi) = \frac{e^{it|\xi|}+e^{-it|\xi|}}{2}\widehat{\phi}_0(\xi) + \frac{e^{it|\xi|}-e^{-it|\xi|}}{2i|\xi|}\widehat{\phi}_1(\xi) - \\ \int \frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(|\xi|-\tau)}\widehat{F}(\tau,\xi)d\tau - \int \frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(\tau+|\xi|)}\widehat{F}(\tau,\xi)d\tau. \tag{1.12}$$

For the homogeneous part, we rewrite it as

$$\frac{e^{it|\xi|}+e^{-it|\xi|}}{2}\widehat{\phi}_0(\xi) + \frac{e^{it|\xi|}-e^{-it|\xi|}}{2i|\xi|}\widehat{\phi}_1(\xi) = \frac{e^{it|\xi|}}{2|\xi|}\widehat{\phi}_+ + \frac{e^{-it|\xi|}}{2|\xi|}\widehat{\phi}_-, \tag{1.13}$$

where

$$\widehat{\phi}_\pm = |\xi|\widehat{\phi}_0 \mp i\widehat{\phi}_1. \tag{1.15}$$

Now we split $\widehat{F}$ the same manner as we did to $\widehat{G}$. Let us write

$$\widehat{F}_\pm(\tau,\xi) := h(\pm\tau)\widehat{a}(\tau\mp|\xi|)\widehat{F}(\tau,\xi), \tag{1.16a}$$

$$\widehat{F}_f(\tau,\xi) := \widehat{F}(\tau,\xi) - \big(\widehat{F}_+(\tau,\xi)+\widehat{F}_-(\tau,\xi)\big), \tag{1.16b}$$

For the inhomogeneous part, we obtain

$$\int \Big[\frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(|\xi|-\tau)} + \frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(|\xi|+\tau)}\Big]\widehat{F}_f(\tau,\xi)d\tau \\ = \int e^{it\tau}\frac{\widehat{F}_f}{|\xi|^2-\tau^2}\,d\tau - \frac{e^{it|\xi|}}{2|\xi|}\int\frac{\widehat{F}_f}{|\xi|-\tau}d\tau - \frac{e^{-it|\xi|}}{2|\xi|}\int\frac{\widehat{F}_f}{|\xi|+\tau}d\tau, \tag{1.17a}$$

$$\int \frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(|\xi|-\tau)}(\widehat{F}_+ + \widehat{F}_-)d\tau = \frac{e^{it|\xi|}}{2|\xi|}\int\frac{e^{it(\tau-|\xi|)}-1}{|\xi|-\tau}(\widehat{F}_+ + \widehat{a}_6\widehat{F}_-)d\tau + \\ \int e^{it\tau}\frac{(1-\widehat{a}_6)\widehat{F}_-}{2|\xi|(|\xi|-\tau)}d\tau - \frac{e^{it|\xi|}}{2|\xi|}\int\frac{(1-\widehat{a}_6)\widehat{F}_-}{|\xi|-\tau}d\tau, \tag{1.17b}$$

$$\int \frac{e^{it\tau} - e^{-it|\xi|}}{2|\xi|(|\xi|+\tau)}(\widehat{F}_+ + \widehat{F}_-)d\tau = \frac{e^{-it|\xi|}}{2|\xi|}\int \frac{e^{it(\tau+|\xi|)}-1}{|\xi|+\tau}(\widehat{a}_6\widehat{F}_+ + \widehat{F}_-)d\tau + \int e^{it\tau}\frac{(1-\widehat{a}_6)\widehat{F}_+}{2|\xi|(|\xi|+\tau)}d\tau - \frac{e^{-it|\xi|}}{2|\xi|}\int \frac{(1-\widehat{a}_6)\widehat{F}_+}{|\xi|+\tau}d\tau. \quad (1.17c)$$

Combining (1.17a)-(1.17c), we can give a formula for $\widehat{\phi}$, namely

$$\widehat{\phi}(\tau,\xi) = \sum_{k=0}^{\infty}\Big(\delta_+^{(k)}(\tau,\xi)\widehat{B}_{+,k}(\xi) + \delta_-^{(k)}(\tau,\xi)\widehat{B}_{-,k}(\xi)\Big) + \widehat{L}(\tau,\xi), \quad (1.18)$$

where $\delta_\pm(\tau,\xi)$ are the delta functions supported on $\{\tau = \pm|\xi|\}$ respectively, $\delta^{(k)}$ mean derivatives of the delta function, and

$$\widehat{L}(\tau,\xi) := \frac{\widehat{F}_f}{\widehat{W}(\tau,\xi)} - \frac{(1-\widehat{a}_6(\tau))\widehat{F}_-}{2|\xi|(|\xi|-\tau)} - \frac{(1-\widehat{a}_6(\tau))\widehat{F}_+}{2|\xi|(|\xi|+\tau)}, \quad (1.19a)$$

$$\widehat{B}_{\pm,0}(\xi) := \frac{1}{2|\xi|}\Big[\widehat{\phi}_\pm + \int \frac{\widehat{F}_f + (1-\widehat{a}_6(\lambda))\widehat{F}_\mp}{|\xi| \mp \lambda}d\lambda\Big], \quad (1.19b)$$

$$\widehat{B}_{\pm,k}(\xi) := \frac{\pm(-1)^k}{2|\xi|k!}\int (\lambda \mp |\xi|)^{k-1}\big[\widehat{F}_\pm + \widehat{a}_6(\lambda)\widehat{F}_\mp\big]d\lambda. \quad (1.19c)$$

Remark. We need to localize the solutions for Dirac equation and wave equation due to the presence of the delta function.

2. Estimates.

To localize the solution in time, let $b(t)$ be a cut-off function such that $b(t)$ equals 1 if $|t| \le \frac{1}{2}$, and equals 0 if $|t| > 1$, and $b_T(t) = b(t/T)$. For an arbitrary function $f(t,x)$, we have

$$\|\widehat{b}_T * \widehat{f}\|_{L^2} = \|b_T f\|_{L^2} \le \|b_T\|_{L^\infty}\|f\|_{L^2}. \quad (2.1)$$

Lemma 2.1. *If $\psi_0 \in H^{-\alpha}$, then we have*

$$\Big\|\widehat{b}_T * [\widehat{M}^{-\alpha}\widehat{S}^{\frac{3}{4}}\widehat{\psi}]\Big\|_{L^2(\mathbb{R}^1\times\mathbb{R}^1)} \le C\Big(\|\psi_0\|_{H^{-\alpha}} + \Big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\Big). \quad (2.2)$$

Proof. Without loss of generality, we prove the special case.

$$\Big\|\widehat{b}_T * [\widehat{S}^{\frac{3}{4}}\widehat{\psi}]\Big\|_{L^2(\mathbb{R}^1\times\mathbb{R}^1)} \le C\Big(\|\psi_0\|_{L^2} + \Big\|\frac{\widehat{G}}{\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\Big). \quad (2.3)$$

To estimate $\widehat{b}_T * [\widehat{S}^{\frac{3}{4}}\widehat{\psi}]$, we apply formulae (1.8) and (1.9)s. First we compute

$$\|\widehat{b}_T * [\widehat{S}^{\frac{3}{4}}\widehat{K}]\|_{L^2} \le \|\widehat{S}^{\frac{3}{4}}\widehat{K}\|_{L^2} \le \Big\|\widehat{S}^{\frac{3}{4}}\frac{\widehat{D}}{\widehat{W}}\widehat{G}_f\Big\|_{L^2} +$$
$$\Big\|\widehat{S}^{\frac{3}{4}}\frac{(1-\widehat{a}_6)\widehat{D}_+\widehat{G}_-}{2|\xi|(\tau-|\xi|)}\Big\|_{L^2} + \Big\|\widehat{S}^{\frac{3}{4}}\frac{(1-\widehat{a}_6)\widehat{D}_-\widehat{G}_+}{2|\xi|(\tau+|\xi|)}\Big\|_{L^2} \le C\Big\|\frac{\widehat{G}}{\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}. \quad (2.4)$$

For the term $\widehat{b}_T * [\widehat{S}^{\frac{3}{4}}\delta_+^{(k)}\widehat{A}_{+,k}]$, we can mollify $\widehat{S}(\tau,\xi)$ without loss of generality such that $\partial_\tau^k\widehat{S}(\pm|\xi|,\xi) = 0$ if $k \ge 1$. Thus we can compute

$$\begin{aligned}
&\|\widehat{b}_T * [\widehat{S}^{\frac{3}{4}}\delta_+^{(k)}](\xi)\|^2_{L^2(d\tau)} \\
\sim &\int\Big(\int \widehat{b}_T(\tau-\lambda)\widehat{S}(\lambda,\xi)^{\frac{3}{4}}\delta^{(k)}(\lambda-|\xi|)d\lambda\Big)^2 d\tau \\
\sim &\int\Big(\frac{\partial^k}{\partial\lambda^k}\big(\widehat{b}_T(\tau-\lambda)\widehat{S}(\lambda,\xi)^{\frac{3}{4}}\big)\Big|_{\lambda=|\xi|}\Big)^2 d\tau \\
\le &\int\Big(T^{k+1}\widehat{b}^{(k)}\big(T(\tau-|\xi|)\big)\Big)^2 d\tau \le T^{2k+1}\|t^k b\|_{L^2} \le CT^{2k+1}.
\end{aligned} \quad (2.5)$$

Then we calculate

$$\begin{aligned}
\|\widehat{A}_{+,0}\|_{L^2(d\xi)} \le &\|\psi_0\|_{L^2} + \Big(\int\Big(\int\frac{\widehat{G}_f + \big(1-\widehat{a}_6(\tau)\big)\widehat{G}_-}{\tau-|\xi|}d\tau\Big)^2 d\xi\Big)^{\frac{1}{2}} \\
\le &\|\psi_0\|_{L^2} + \Big\|\frac{\widehat{G}}{\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}
\end{aligned} \quad (2.6)$$

and

$$\begin{aligned}
\|\widehat{A}_{+,k}\|_{L^2(d\xi)} \le &\frac{1}{k!}\Big(\int\Big(\int(\tau-|\xi|)^{k-1}\big[\widehat{G}_+ + \widehat{a}_6\widehat{G}_-\big](\tau,\xi)d\tau\Big)^2 d\xi\Big)^{\frac{1}{2}} \\
\le &\frac{2^k}{k!}\Big(\int\int\Big|\widehat{G}_+ + \widehat{a}_6\widehat{G}_-\Big|^2(\tau,\xi)d\tau d\xi\Big)^{\frac{1}{2}} \le \frac{2^k}{k!}\Big\|\frac{\widehat{G}}{\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}.
\end{aligned} \quad (2.7)$$

Therefore we have

$$\begin{aligned}
\|\widehat{b}_T * [\widehat{S}^{\frac{3}{4}}\delta_+\widehat{A}_{+,0}]\|_{L^2} \le &T^{\frac{1}{2}}\Big(\|\psi_0\|_{L^2} + \Big\|\frac{\widehat{G}}{\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\Big), \\
\|\widehat{b}_T * [\widehat{S}^{\frac{3}{4}}\delta_+^{(k)}\widehat{A}_{+,k}]\|_{L^2} \le &T^{k+\frac{1}{2}}\frac{2^k}{k!}\Big\|\frac{\widehat{G}}{\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}.
\end{aligned} \quad (2.8)$$

The calculation for the term $\widehat{b}_T * [\widehat{S}^{\frac{3}{4}} \delta_-^{(k)} \widehat{A}_{-,k}]$ is analogous. Combine the above results we complete the proof. □

Consider two Dirac equations,

$$\begin{cases} \mathcal{D}\psi_j = G_j, \quad j = 1, 2, \\ \psi_j(0) = \psi_{0j}. \end{cases} \tag{2.9}$$

For the solutions of (2.9), we have the following key estimate whose proof will be presented in the last section.

Lemma 2.2. *(Null Form Estimate) Let* $\alpha = \frac{1}{4} - \epsilon$, $\epsilon > 0$, *and* ψ_1, ψ_2 *be the solutions for (2.9). If* $\psi_{0j} \in H^{-\alpha}$, *we have*

$$\Big\| \frac{\widehat{(b_T \overline{\psi}_1 \psi_2)}}{\widehat{E}^\alpha \widehat{S}^\alpha} \Big\|_{L^2} \leq C(T) \Big(\|\psi_{01}\|_{H^{-\alpha}} + \Big\| \frac{\widehat{G}_1}{\widehat{M}^\alpha \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2} \Big) \cdot \Big(\|\psi_{02}\|_{H^{-\alpha}} + \Big\| \frac{\widehat{G}_2}{\widehat{M}^\alpha \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2} \Big). \tag{2.10}$$

For the wave equation (1.10), we have the following estimate.

Lemma 2.3. *Let* ϕ *be the solution of (1.10). If* $\phi_0 \in H^{1-2\alpha}$ *and* $\phi_1 \in H^{-2\alpha}$, *then*

$$\|\widehat{b}_T * \Big[\widehat{M}^{-\alpha} (\widehat{E}\widehat{S})^{1-\alpha} \widehat{\phi} \Big] \|_{L^2} \leq C \Big(\|\phi_0\|_{H^{1-2\alpha}} + \|\phi_1\|_{H^{-2\alpha}} + \Big\| \frac{\widehat{F}}{\widehat{M}^\alpha \widehat{E}^\alpha \widehat{S}^\alpha} \Big\|_{L^2} \Big). \tag{2.11}$$

Proof. Without loss of generality, we show the following special case:

$$\|\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha} \widehat{\phi} \big] \|_{L^2} \leq C \Big(\|\phi_0\|_{H^{1-\alpha}} + \|\phi_1\|_{H^{-\alpha}} + \Big\| \frac{\widehat{F}}{\widehat{E}^\alpha \widehat{S}^\alpha} \Big\|_{L^2} \Big). \tag{2.12}$$

To estimate $\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha} \widehat{\phi} \big]$ in the L^2-norm, we invoke the formulae (1.18) and (1.19). First we compute

$$\|\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha} \widehat{L} \big] \|_{L^2} \leq \|(\widehat{E}\widehat{S})^{1-\alpha} \widehat{L}\|_{L^2} \leq \Big\| \frac{(\widehat{E}\widehat{S})^{1-\alpha} \widehat{F}_f}{\widehat{W}} \Big\|_{L^2} + \Big\| \frac{(\widehat{E}\widehat{S})^{1-\alpha} (1 - \widehat{a}_6) \widehat{F}_-}{2|\xi|(|\xi| - \tau)} \Big\|_{L^2} + \Big\| \frac{(\widehat{E}\widehat{S})^{1-\alpha} (1 - \widehat{a}_6) \widehat{F}_+}{2|\xi|(|\xi| + \tau)} \Big\|_{L^2} \leq \Big\| \frac{\widehat{F}}{\widehat{E}^\alpha \widehat{S}^\alpha} \Big\|_{L^2}. \tag{2.13}$$

For the term $\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}\delta_+^{(k)}\widehat{B}_{+,k}\big]$, we can mollify $\widehat{E}\widehat{S}(\tau,\xi)$ without loss of generality such that $\partial_\tau^k \widehat{S}(\pm|\xi|,\xi) = 0$ if $k \geq 1$. Thus we compute

$$\begin{aligned}
&\big\|\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}\delta_+^{(k)}\big](\xi)\big\|^2_{L^2(d\tau)} \\
=& \int \Big| \int \widehat{b}_T(\tau-\lambda)(\widehat{E}\widehat{S})^{1-\alpha}(\lambda,\xi)\delta^{(k)}(\lambda-|\xi|)d\lambda\Big|^2 d\tau \\
=& \int \Big| \frac{\partial^k}{\partial\lambda^k}\Big(\widehat{b}_T(\tau-\lambda)(\widehat{E}\widehat{S})^{1-\alpha}(\lambda,\xi)\Big)\Big|_{\lambda=|\xi|}\Big|^2 d\tau \\
\sim& \int \Big| T^{k+1}\widehat{b}^{(k)}\big(T(\tau-|\xi|)\big)\Big|^2 (|\xi|+1)^{2-2\alpha} d\tau \\
\leq& \, T^{2k+1}\|t^k b\|_{L^2}(|\xi|+1)^{2-2\alpha} \leq \; CT^{2k+1}(|\xi|+1)^{2-2\alpha},
\end{aligned} \tag{2.14}$$

which implies

$$\big\|\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}\delta_+^{(k)}\widehat{B}_{+,k}\big]\big\|_{L^2} \leq CT^{k+\frac{1}{2}}\Big(\int (|\xi|+1)^{2-2\alpha}\big|\widehat{B}_{+,k}(\xi)\big|^2 d\xi\Big)^{\frac{1}{2}}. \tag{2.15}$$

To estimate the above integral, we first focus on the region where $|\xi| > 1$. Due to the observation that on the supports of $(1-\widehat{a}_6)\widehat{F}_-$ and $\widehat{F}_f$, the following inequality holds

$$\widehat{E}^{2\alpha}\widehat{S}^{2\alpha} = \big(|\lambda|+|\xi|+1\big)^{2\alpha}\big(\big||\lambda|-|\xi|\big|+1\big)^{2\alpha} \leq (|\xi|+1)^{2\alpha}\big|\lambda-|\xi|\big|^{4\alpha}, \tag{2.16}$$

we have the following bounds:

$$\begin{aligned}
&\int \Big| \int \frac{\widehat{F}_f(\lambda,\xi)}{(|\xi|+1)^\alpha\big||\xi|-\lambda\big|} d\lambda\Big|^2 d\xi \\
\leq& \int \int_{||\xi|-\lambda|\geq\frac{1}{2}} \frac{1}{\big||\xi|-\lambda\big|^{1+4\epsilon}} d\lambda \int \frac{\big|\widehat{F}_f(\lambda,\xi)\big|^2}{(|\xi|+1)^{2\alpha}\big||\xi|-\lambda\big|^{1-4\epsilon}} d\lambda\, d\xi \\
\leq& C\Big\|\frac{\widehat{F}_f}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|^2_{L^2}
\end{aligned} \tag{2.17}$$

and in the same vein

$$\int \Big| \int \frac{\big(1-\widehat{a}_6(\lambda)\big)\widehat{F}_-(\lambda,\xi)}{(|\xi|+1)^\alpha\big||\xi|-\lambda\big|} d\lambda\Big|^2 d\xi \leq C\Big\|\frac{\widehat{F}_-}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|^2_{L^2}. \tag{2.18}$$

Hence we get

$$\begin{aligned}&\big\|\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}\delta_+\widehat{B}_{+,0}\big]\big\|_{L^2(L^2(|\xi|>1))}\\ &\le CT^{\frac{1}{2}}\Big(\|\phi_0\|_{H^{1-\alpha}} + \|\phi_1\|_{H^{-\alpha}} + \Big\|\frac{\widehat{F}}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2}\Big)\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}&\big\|\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}\delta_+^{(k)}\widehat{B}_{+,k}\big]\big\|_{L^2(L^2(|\xi|>1))}\\ &\le CT^{k+\frac{1}{2}}\frac{c^k}{k!}\Big(\int\int\frac{\big|\widehat{F}_+ + \widehat{a}_6\widehat{F}_-\big|^2(\tau,\xi)}{(|\xi|+1)^{2\alpha}}d\lambda\, d\xi\Big)^{\frac{1}{2}}\\ &\le CT^{k+\frac{1}{2}}\frac{c^k}{k!}\Big\|\frac{\widehat{F}}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2}.\end{aligned} \tag{2.20}$$

The calculation for the term $\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}\delta_-^{(k)}\widehat{B}_{-,k}\big]$ is analogous.

For the region $|\xi|\le 1$, we consider $\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}(\delta_+^{(k)}\widehat{B}_{+,k} + \delta_-^{(k)}\widehat{B}_{-,k})\big]$. This is clear from the derivation of the solution representation which indicates that the solution is actually not singular along the cones.

$$\begin{aligned}&\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}(\delta_+^{(k)}\widehat{B}_{+,k} + \delta_-^{(k)}\widehat{B}_{-,k})\big](\tau,\xi)\\ &\sim T^{k+1}\big(|\xi|+1\big)^{1-\alpha}\big[\widehat{t^kb}\big(T(\tau-|\xi|)\big)\widehat{B}_{+,k}(\xi) + \widehat{t^kb}\big(T(\tau+|\xi|)\big)\widehat{B}_{-,k}(\xi)\big]\\ &= T^{k+1}\big(|\xi|+1\big)^{1-\alpha}\big[\widehat{t^kb}\big(T(\tau-|\xi|)\big) - \widehat{t^kb}\big(T(\tau+|\xi|)\big)\big]\widehat{B}_{+,k}(\xi) +\\ &\quad T^{k+1}\big(|\xi|+1\big)^{1-\alpha}\widehat{t^kb}\big(T(\tau+|\xi|)\big)\big[\widehat{B}_{+,k}(\xi) + \widehat{B}_{-,k}(\xi)\big].\end{aligned} \tag{2.21}$$

Under the restriction of $|\xi|\le 1$, we have

$$\widehat{t^kb}\big(T(\tau-|\xi|)\big) - \widehat{t^kb}\big(T(\tau+|\xi|)\big) \sim T\widehat{t^{k+1}b}\big(T(\tau-(1-2\theta)|\xi|)\big)|\xi|, \tag{2.23}$$

$$\widehat{B}_{+,0}(\xi) + \widehat{B}_{-,0}(\xi) \sim \widehat{\phi}_0 + \int\frac{\widehat{F}_f}{|\xi|^2-\lambda^2}\,d\lambda, \tag{2.24}$$

and

$$\widehat{B}_{+,k}(\xi) + \widehat{B}_{-,k}(\xi) \sim \frac{1}{(k-1)!}\int\big(\lambda-(1-2\theta)|\xi|\big)^{k-2}(\widehat{F}_+ + \widehat{a}_6\widehat{F}_-)\,d\lambda. \tag{2.25}$$

Combine the above results we complete the proof. □

We will also need some technical lemmas.

Lemma 2.4. *(Hardy-Littlewood-Polya) Let* $r = 2 - \frac{1}{p} - \frac{1}{q}$. *Then we have*

$$\int_{\mathbb{R}^1\times\mathbb{R}^1} \frac{f(s)g(t)}{|s-t|^r} dsdt \le C\|f\|_{L^p}\|g\|_{L^q}. \tag{2.26}$$

Lemma 2.5. *Let* $f(t,x)$ *and* $g(t,x)$ *be any functions such that* $f \in L^q(L^2(\mathbb{R}))$ *and* $\widehat{S}^\beta \widehat{g} \in L^2(L^2(\mathbb{R}))$. *Assume that* $\delta \ge 0$, $q = \frac{8}{5-4\delta}$, $\frac{1}{r} = \frac{1}{2} - \beta$, *and* $2 \le r < \infty$. *Then we have*

$$\Big\|\frac{\widehat{b_T} * \widehat{f}}{\widehat{S}^{\frac{1}{4}-\delta}}\Big\|_{L^2} \le C\|b_T f\|_{L^q(L^2)}, \tag{2.27}$$

$$\|g\|_{L^r(L^2)} \le C\|\widehat{S}^\beta \widehat{g}\|_{L^2(L^2)}. \tag{2.28}$$

Proof. The proofs for (2.27) and (2.28) are analogous. Therefore we will only prove the case of (2.28).

Taking the inverse Fourier transform in the time variable over the identity

$$\widehat{g} = \frac{1}{\widehat{S}^\beta}\widehat{S}^\beta\widehat{g} \tag{2.29}$$

gives

$$\widetilde{g}(t,\xi) = \int \frac{e^{\pm i(t-s)|\xi|}}{|t-s|^{1-\beta}} \mathcal{F}_\tau^{-1}(\widehat{S}^\beta\widehat{g})(s,\xi)\, ds. \tag{2.30}$$

Then we use duality and Hardy-Littlewood-Polya inequality to compute

$$\begin{aligned}\Big|\Big\langle g,\varphi\Big\rangle\Big| =& \Big|\Big\langle \widetilde{g},\widetilde{\varphi}\Big\rangle\Big| = \Big|\iint\int \frac{e^{\pm i(t-s)|\xi|}}{|t-s|^{1-\beta}} \mathcal{F}_\tau^{-1}(\widehat{S}^\beta\widehat{g})(s,\xi)\, ds\overline{\widetilde{\varphi}}(t,\xi)\, dtd\xi\Big| \\ \le& \int \frac{\|\mathcal{F}_\tau^{-1}(\widehat{S}^\beta\widehat{g})(s)\|_{L^2}\|\widetilde{\varphi}(t)\|_{L^2}}{|t-s|^{1-\beta}} dsdt \\ \le& C\|\mathcal{F}_\tau^{-1}(\widehat{S}^\beta\widehat{g})\|_{L^2}\|\widetilde{\varphi}\|_{L^{r'}(L^2)} = C\|\widehat{S}^\beta\widehat{g}\|_{L^2}\|\varphi\|_{L^{r'}(L^2)}.\end{aligned} \tag{2.31}$$

This completes the proof of (2.28). □

3. Local Existence.

Now we are ready to prove the local existence for the (DKG) equations.

Proof of Theorem 0.1. Consider the DKG problem

$$\mathcal{D}\psi = b_T\phi\psi; \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^1 \tag{3.1a}$$

$$\Box\phi = b_T\overline{\psi}\psi; \tag{3.1b}$$

$$\psi(0,x) := \psi_0(x), \quad \phi(0,x) = \phi_0(x), \quad \phi_t(0,x) = \phi_1(x), \tag{3.1c}$$

Iteration scheme induces a map $\mathcal{T}$ defined by

$$\mathcal{T}(\psi^k, \phi^k) = (\psi^{k+1}, \phi^{k+1}). \tag{3.2a}$$

We want to show that $\mathcal{T}$ is a contraction under the norm

$$\mathcal{N}(\psi,\phi) = \big\|\widehat{M}^{-\alpha}\widehat{S}^{\frac{3}{4}}\widehat{\psi}\big\|_{L^2} + \big\|\widehat{M}^{-\alpha}(\widehat{E}\widehat{S})^{1-\alpha}\widehat{\phi}\big\|_{L^2}. \tag{3.2b}$$

For convenience, we call

$$J(0) = \|\phi_0\|_{H^{1-2\alpha}} + \|\phi_1\|_{H^{-2\alpha}} + \|\psi_0\|^2_{H^{-\alpha}} + 1. \tag{3.3}$$

First we apply (2.11), (2.10), and (2.27) to compute

$$\begin{aligned}\big\|\widehat{M}^{-\alpha}(\widehat{E}\widehat{S})^{1-\alpha}\widehat{\mathcal{T}\phi}\big\|_{L^2} &\le C\Big(J(0) + \Big\|\frac{\widehat{b_T\overline{\psi}\psi}}{\widehat{M}^\alpha\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2}\Big)\\ &\le C\Big(J(0) + \Big\|\frac{\widehat{b_T\phi\psi}}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{4}}}\Big\|^2_{L^2}\Big)\\ &\le C\Big(J(0) + \Big\|\frac{b_T\widetilde{\phi\psi}}{\widehat{M}^\alpha}\Big\|^2_{L^{\frac{8}{5}}([0,T],L^2)}\Big)\\ &\le C\Big(J(0) + T^{\frac{1}{8}}\Big\|\frac{b_T\widetilde{\phi\psi}}{\widehat{M}^\alpha}\Big\|^2_{L^2([0,T],L^2)}\Big).\end{aligned} \tag{3.4}$$

To bound the term above, we first compute

$$\begin{aligned}\Big\|\widehat{M}^{-\alpha}\widetilde{\phi\psi}(t)\Big\|_{L^2} &\sim \|G_\alpha * (\phi\psi)(t)\|_{L^2}\\ &\le \|\phi(t)\|_{L^\infty}\|G_\alpha * \psi(t)\|_{L^2} \le \|\phi(t)\|_{H^{1-2\alpha}}\|\psi(t)\|_{H^{-\alpha}},\end{aligned} \tag{3.5}$$

where $G_\alpha(x)$ is an L^1-function with the following property:

$$\widehat{G_\alpha}(\xi) \sim (1+|\xi|)^{-\alpha}, \tag{3.6}$$

see [S], then we invoke (2.27) and (2.28) to obtain

$$\begin{aligned}
\left\| b_T \widehat{M}^{-\alpha} \widetilde{\widehat{\phi}\widehat{\psi}} \right\|_{L^2} &\le C\|\phi\|_{L^4([0,T],H^{1-2\alpha})}\|\psi\|_{L^4([0,T],H^{-\alpha})} \\
&\le C\left\|\widehat{S}^{\frac14}\widehat{M}^{1-2\alpha}\widehat{\phi}\right\|_{L^2}\left\|\widehat{S}^{\frac14}\widehat{M}^{-\alpha}\widehat{\psi}\right\|_{L^2} \\
&\le C\left\|\widehat{M}^{\alpha}\widehat{S}^{\frac14}\widehat{M}^{1-\alpha}\widehat{\phi}\right\|_{L^2}\left\|\widehat{S}^{\frac34}\widehat{M}^{-\alpha}\widehat{\psi}\right\|_{L^2} \\
&\le C\left\|\widehat{M}^{-\alpha}(\widehat{E}\widehat{S})^{1-\alpha}\widehat{\phi}\right\|_{L^2}\left\|\widehat{M}^{-\alpha}\widehat{S}^{\frac34}\widehat{\psi}\right\|_{L^2}.
\end{aligned} \tag{3.7}$$

Next we want to bound the term involved with $\widehat{\psi}$. The estimate (2.2) implies that

$$\left\|\widehat{M}^{-\alpha}\widehat{S}^{\frac34}\widehat{\mathcal{T}\psi}\right\|_{L^2} \le C\Big(\|\psi_0\|_{H^{-\alpha}} + \Big\|\frac{\widehat{b_T\phi\psi}}{\widehat{M}^{\alpha}\widehat{S}^{\frac14}}\Big\|_{L^2}\Big). \tag{3.9}$$

Hence, using (3.4), (3.7), and (3.9), we have

$$\mathcal{N}(\mathcal{T}(\psi,\phi)) \le C\big(J(0) + T^{\frac18}N^4(\psi,\phi)\big). \tag{3.10}$$

Choosing sufficiently large L, for suitable T, we have

$$\mathcal{N}(\psi,\phi) \le L \Longrightarrow N\big(\mathcal{T}(\psi,\phi)\big) \le L, \tag{3.11}$$

provided that

$$C(J(0) + T^{\frac18}L^4) \le L. \tag{3.12}$$

Now we consider the difference $\mathcal{T}(\psi,\phi) - \mathcal{T}(\psi',\phi')$. Base on the observations

$$\overline{\psi}\psi - \overline{\psi'}\psi' = \frac12\overline{(\psi-\psi')}(\psi+\psi') + \frac12\overline{(\psi+\psi')}(\psi-\psi'), \tag{3.13a}$$

$$\phi\psi - \phi'\psi' = \frac12(\phi-\phi')(\psi+\psi') + \frac12(\phi+\phi')(\psi-\psi'), \tag{3.13b}$$

Employing (2.11), (2.10), and (3.13), we first calculate

$$
\begin{aligned}
&\big\|\widehat{M}^{-\alpha}(\widehat{E}\widehat{S})^{1-\alpha}\mathcal{F}(\mathcal{T}\phi-\mathcal{T}\phi')\big\|_{L^2}\\
\le& C\Big(\|\frac{\mathcal{F}(b_T(\overline{\psi-\psi'})(\psi+\psi'))}{\widehat{M}^{\alpha}\widehat{E}^{\alpha}\widehat{S}^{\alpha}}\|_{L^2}+\|\frac{\mathcal{F}(b_T(\overline{\psi+\psi'})(\psi-\psi'))}{\widehat{M}^{\alpha}\widehat{E}^{\alpha}\widehat{S}^{\alpha}}\|_{L^2}\Big)\\
\le& C\Big(\|\frac{\mathcal{F}(b_T(\phi-\phi')(\psi+\psi'))}{\widehat{M}^{\alpha}\widehat{S}^{\frac14}}\|_{L^2}+\|\frac{\mathcal{F}(b_T(\phi+\phi')(\psi-\psi'))}{\widehat{M}^{\alpha}\widehat{S}^{\frac14}}\|_{L^2}\Big)\cdot\\
&\Big(I.D.+\|\frac{\mathcal{F}(b_T(\phi\psi+\phi'\psi'))}{\widehat{M}^{\alpha}\widehat{S}^{\frac14}}\|_{L^2}\Big)\\
\le& CT^{\frac18}\big(\|\widehat{M}^{-\alpha}(\widehat{E}\widehat{S})^{1-\alpha}\widehat{\phi-\phi'}\|_{L^2}+\|\widehat{M}^{-\alpha}\widehat{S}^{\frac34}\widehat{\psi-\psi'}\|_{L^2}\big)L(I.D.+L^2)\\
\le& CT^{\frac18}L^3\big(\|\widehat{M}^{-\alpha}(\widehat{E}\widehat{S})^{1-\alpha}\widehat{\phi-\phi'}\|_{L^2}+\|\widehat{M}^{-\alpha}\widehat{S}^{\frac34}\widehat{\psi-\psi'}\|_{L^2}\big)
\end{aligned}
\tag{3.15}
$$

Analogously, we get

$$
\begin{aligned}
&\big\|\widehat{M}^{-\alpha}\widehat{S}^{\frac34}\mathcal{F}(\mathcal{T}\psi-\mathcal{T}\psi')\big\|_{L^2}\\
&\quad\le CT^{\frac18}L\Big(\|\widehat{M}^{-\alpha}(\widehat{E}\widehat{S})^{1-\alpha}\widehat{\phi-\phi'}\|_{L^2}+\|\widehat{M}^{-\alpha}\widehat{S}^{\frac34}\widehat{\psi-\psi'}\|_{L^2}\Big).
\end{aligned}
\tag{3.16}
$$

Combining (3.15)and (3.16), we have

$$
\mathcal{N}\big(\mathcal{T}(\psi-\psi',\phi-\phi')\big)\le CT^{\frac18}L^3N(\psi-\psi',\phi-\phi'). \tag{3.17}
$$

Therefore for suitable T, we obtain

$$
\mathcal{N}\big(\mathcal{T}(\psi-\psi',\phi-\phi')\big)\le \frac{1}{2}N(\psi-\psi',\phi-\phi'), \tag{3.18}
$$

provided that

$$
CT^{\frac18}L^3\le\frac12. \tag{3.19}
$$

We can conclude that the map $\mathcal{T}$ is indeed a contraction with respect to the norm $\mathcal{N}$, thus it has a unique fixed point. □

We now prove the global existence.

Proof of Theorem 0.2. From the law of conservation of charge, we have

$$
\sup_{[0,T]}\|\psi(t)\|_{L^2}=\|\psi_0\|_{L^2}. \tag{3.20}
$$

To bound ϕ we apply the following formula,

$$2\phi(t,x) = \phi_0(x+t)+\phi_0(x-t)+\int_{x-t}^{x+t}\phi_1(y)\,dy+\int_0^t\int_{x-t+s}^{x+t-s}\overline{\psi}\psi(s,y)dyds. \tag{3.21}$$

First we write $\phi = \phi_L + \phi_N$, the homogeneous and inhomogeneous parts of the solution, then we obtain

$$\begin{aligned}\|\phi_L(t)\|_{L^\infty} \leq&\|\phi_L(t)\|_{H^{\frac{1}{2}+\delta}}\\ \leq&\|\phi_0\|_{H^{\frac{1}{2}+\delta}}+\|\phi_1\|_{H^{-\frac{1}{2}+\delta}}\leq J(0),\end{aligned} \tag{3.22}$$

and

$$\|\phi_N(t)\|_{L^\infty}\leq\int_0^t\int_{x-t+s}^{x+t-s}\left|\overline{\psi}\psi(s,y)\right|dyds\leq CT\|\psi_0\|_{L^2}^2. \tag{3.23}$$

Combine (3.22) and (3.23), we get

$$\|\phi(t)\|_{L^\infty}\leq C(T,J(0)). \tag{3.24}$$

Take Fourier transform of the solution $\phi(t)$, we have

$$\widetilde{\phi}(t,\xi)=\cos t|\xi|\widehat{\phi}_0(\xi)+\frac{\sin t|\xi|}{|\xi|}\widehat{\phi}_1(\xi)+\int_0^t\frac{\sin{(t-s)|\xi|}}{|\xi|}\widetilde{\overline{\psi}\psi}(s,\xi)ds. \tag{3.25}$$

Then we invoke (3.21), (2.10) (for $\alpha=0$), (2.27), and (3.24) to compute

$$\begin{aligned}\|\phi(t)\|_{H^{\frac{1}{2}+\delta}}\leq&\|\phi_0\|_{H^{\frac{1}{2}+\delta}}+\|\phi_1\|_{H^{-\frac{1}{2}+\delta}}+\int_0^t\|b_T\overline{\psi}\psi(s)\|_{H^{-\frac{1}{2}+\delta}}ds\\ \leq&J(0)+T^{\frac{1}{2}}\Big\|\frac{\widehat{b_T\overline{\psi}\psi}}{\widehat{M}^{\frac{1}{2}-\delta}}\Big\|_{L^2}\\ \leq&J(0)+T^{\frac{1}{2}}\|\widehat{b_T\overline{\psi}\psi}\|_{L^2}\\ \leq&J(0)+T^{\frac{1}{2}}\Big\|\frac{\widehat{b_T\phi\psi}}{\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}^2\\ \leq&J(0)+T^{\rho}\|b_T\phi\psi\|_{L^2}^2\\ \leq&J(0)+T^{\rho}\int_0^T\|\phi(t)\|_{L^\infty}^2\|\psi(t)\|_{L^2}^2dt\\ \leq&C\big(T,J(0)\big),\end{aligned} \tag{3.26}$$

where ρ is some positive number. The calculation for $\|\phi_t(t)\|_{H^{-\frac{1}{2}+\delta}}$ is analogous. Thus the above bounds ensure us to proceed the construction of solution beyond T. □

4. Null Form Estimate.

In this section, we demonstrate the proof of the key estimate.

Lemma 2.2. *(Null Form Estimate) Let* $\alpha = \frac{1}{4} - \epsilon$, $\epsilon > 0$, *and* ψ_1, ψ_2 *be the solutions for the Dirac equations (2.9). If the initial data* $\psi_{0j} \in H^{-\alpha}$, $j = 1, 2$, *then we have*

$$\left\| \frac{\widehat{b_T \overline{\psi}_1 \psi_2}}{\widehat{E}^\alpha \widehat{S}^\alpha} \right\|_{L^2} \leq C(T)\Big(\|\psi_{01}\|_{H^{-\alpha}} + \left\| \frac{\widehat{G}_1}{\widehat{M}^\alpha \widehat{S}^{\frac{1}{4}}} \right\|_{L^2} \Big) \cdot \Big(\|\psi_{02}\|_{H^{-\alpha}} + \left\| \frac{\widehat{G}_2}{\widehat{M}^\alpha \widehat{S}^{\frac{1}{4}}} \right\|_{L^2} \Big). \tag{4.1}$$

The proof for the estimate is based on the duality argument and it will be given in a number of steps. Without loss of generality, we assume that $\psi_1 = \psi_2$, and prove: if ψ is a solution of the Dirac equation (1.2), then

$$\left\| \frac{\widehat{b_T \overline{\psi} \psi}}{\widehat{E}^\alpha \widehat{S}^\alpha} \right\|_{L^2} \leq C(T)\Big(\|\psi_0\|_{H^{-\alpha}} + \left\| \frac{\widehat{G}}{\widehat{M}^\alpha \widehat{S}^{\frac{1}{4}}} \right\|_{L^2} \Big)^2. \tag{4.2}$$

Recall that the notations:

$$\widehat{E}(\tau,\xi) := |\tau| + |\xi| + 1, \qquad \widehat{S}(\tau,\xi) := \Big| |\tau| - |\xi| \Big| + 1, \tag{4.3a}$$

$$\widehat{W}(\tau,\xi) := \tau^2 - |\xi|^2, \qquad \widehat{D}(\tau,\xi) := \gamma^0 \tau + \gamma^1 \xi, \tag{4.3b}$$

$$\widehat{D}_+ := \widehat{D}(|\xi|, +\xi), \qquad \widehat{D}_- := \widehat{D}(|\xi|, -\xi). \tag{4.3c}$$

The formula for $\widehat{\psi}$, as in (1.8), for the Dirac equation (1.2) is given by

$$\widehat{\psi}(\tau,\xi) = \sum_{k=0}^{\infty} \Big(\delta_+^{(k)}(\tau,\xi) \widehat{A}_{+,k}(\xi) + \delta_-^{(k)}(\tau,\xi) \widehat{A}_{-,k}(\xi) \Big) + \widehat{K}(\tau,\xi), \tag{4.4}$$

where $\delta_\pm(\tau,\xi)$ are the delta functions supported on $\{\tau=\pm|\xi|\}$ respectively, $\delta^{(k)}$ mean derivatives of the delta function, and

$$\widehat{K}(\tau,\xi):=\frac{\widehat{D}(\tau,\xi)}{\widehat{W}(\tau,\xi)}\widehat{G}_f+\frac{(1-\widehat{a}_6(\tau))\widehat{D}_+\widehat{G}_-}{2|\xi|(\tau-|\xi|)}+\frac{(1-\widehat{a}_6)\widehat{D}_-\widehat{G}_+}{2|\xi|(\tau+|\xi|)}, \tag{4.5a}$$

$$\widehat{A}_{\pm,0}(\xi):=\frac{\widehat{D}_\pm}{2|\xi|}\Big[\gamma^0\widehat{\psi}_0-\int\frac{\widehat{G}_f+(1-\widehat{a}_6(\lambda))\widehat{G}_\mp}{\lambda\mp|\xi|}d\lambda\Big], \tag{4.5b}$$

$$\widehat{A}_{\pm,k}(\xi):=\frac{\widehat{D}_\pm(-1)^k}{2|\xi|k!}\int(\lambda\mp|\xi|)^{k-1}\big[\widehat{G}_\pm+\widehat{a}_6(\lambda)\widehat{G}_\mp\big]d\lambda. \tag{4.5c}$$

Moreover we write

$$\widehat{A}_{\pm,k}(\xi):=\frac{\widehat{D}_\pm}{2|\xi|}\widehat{f}_{\pm,k}(\xi), \tag{4.6}$$

and split $\widehat{K}=\widehat{K}_1+\widehat{K}_2$, where

$$\widehat{K}_1:=\frac{\widehat{D}(\tau,\xi)}{\widehat{W}(\tau,\xi)}\widehat{G}_f;\quad \widehat{K}_2:=\frac{b_1\widehat{D}_+\widehat{G}_-+b_2\widehat{D}_-\widehat{G}_+}{\widehat{E}\widehat{S}}, \tag{4.7}$$

and b_1, b_2 are bounded functions. The Fourier transform of the quadratic expression, $\widehat{\overline{\psi}\psi}=\widehat{\overline{\psi}}*\widehat{\psi}$, can be written as the sum of the following terms.

$$\sum_{k,l}\big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big)*\big(\delta_\pm^{(l)}\widehat{A}_{\pm,l}\big), \tag{4.8a}$$

$$\sum_{k,l}\big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big)*\big(\delta_\mp^{(l)}\widehat{A}_{\mp,l}\big), \tag{4.8b}$$

$$\sum_{k}\big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big)*\big(\widehat{K}_1+\widehat{K}_2\big)+\big(\widehat{\overline{K}}_1+\widehat{\overline{K}}_2\big)*\sum_{k}\big(\delta_\pm^{(k)}\widehat{A}_{\pm,k}\big), \tag{4.8c}$$

$$\widehat{\overline{K}}_1*\widehat{K}_1+\widehat{\overline{K}}_1*\widehat{K}_2+\widehat{\overline{K}}_2*\widehat{K}_1+\widehat{\overline{K}}_2*\widehat{K}_2. \tag{4.8d}$$

Notice that

$$\widehat{A^\dagger_{\pm,k}}(\xi)=\widehat{A}^\dagger_{\pm,k}(-\xi);\qquad \widehat{f^\dagger_{\pm,k}}(\xi)=\widehat{f}^\dagger_{\pm,k}(-\xi), \tag{4.9a}$$

$$\widehat{\overline{A}_{\pm,k}}(\xi)=\widehat{f}^\dagger_{\pm,k}(-\xi)\frac{\widehat{D}_\pm}{|\xi|}\gamma^0;\quad \widehat{\overline{K}}(\tau,\xi)=\widehat{K}^\dagger(-\tau,-\xi)\gamma^0, \tag{4.9b}$$

and

$$\widehat{\overline{\psi}}(\tau,\xi)=\sum_{k=0}^{\infty}\Big(\delta_-^{(k)}(\tau,\xi)\widehat{\overline{A}}_{+,k}(\xi)+\delta_+^{(k)}(\tau,\xi)\widehat{\overline{A}}_{-,k}(\xi)\Big)+\widehat{\overline{K}}(\tau,\xi), \tag{4.10}$$

Lemma 4.1. *Let $\alpha < \frac{1}{4}$. The following estimate holds*

$$\Big\| \frac{\widehat{b}_T * \big(\delta_{\mp}^{(k)} \widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_{\mp}^{(l)} \widehat{A}_{\mp,l}\big)}{\widehat{E}^{\alpha} \widehat{S}^{\alpha}} \Big\|_{L^2} \\ \leq C(k+l+1) T^{k+l-2\alpha} \|f_{\pm,k}\|_{H^{-\alpha}} \|f_{\mp,l}\|_{H^{-\alpha}}. \quad (4.11)$$

Proof. Let us call

$$\widehat{Z}_{\pm,k} \equiv \delta_{\pm}^{(k)} \widehat{A}_{\pm,k} = \delta_{\pm}^{(k)} \frac{\widehat{D}_{\pm}}{|\xi|} \widehat{f}_{\pm,k}. \quad (4.12)$$

Using duality, we demonstrate the case $(-,+)$, while the case $(+,-)$ is being similar. We first compute the fractional term

$$\frac{\widehat{D}(|\xi|,-\xi)\gamma^0 \widehat{D}(|\eta|,\eta)}{|\xi||\eta|} = \begin{cases} 0, & \text{if } \xi\eta > 0, \\ 2\gamma^0 \pm 2\gamma^1, & \text{if } \xi\eta < 0, \end{cases} \quad (4.13)$$

and observe that, for $\xi\eta < 0$,

$$\big||\xi| + |\eta|\big| + |\xi+\eta| + 1 \sim \max\{|\xi|,|\eta|\} + 1, \quad (4.14a)$$

$$\Big|\big||\xi| + |\eta|\big| - |\xi+\eta|\Big| + 1 \sim \min\{|\xi|,|\eta|\} + 1. \quad (4.14b)$$

Thus

$$\begin{aligned} &\Big| \big\langle b_T \overline{Z}_{-,k} Z_{+,l}, g \big\rangle \Big| \\ =&\Big| \int \widehat{f}^{\dagger}_{-,k}(-\xi) \frac{\widehat{D}(|\xi|,-\xi)\gamma^0 \widehat{D}(|\eta|,\eta)}{|\xi||\eta|} \widehat{f}_{+,l}(\eta) \overline{\widehat{t^{k+l} b_T g}}(|\xi|+|\eta|, \xi+\eta)\, d\xi d\eta \Big| \\ \leq& C \|f_{-,k}\|_{H^{-\alpha}} \|f_{+,l}\|_{H^{-\alpha}} \cdot \\ &\Big(\int (|\xi|+1)^{2\alpha} (|\eta|+1)^{2\alpha} \big| \widehat{t^{k+l} b_T g}(|\xi|+|\eta|, \xi+\eta) \big|^2 \, d\xi d\eta \Big)^{\frac{1}{2}} \\ \leq& C \|f_{-,k}\|_{H^{-\alpha}} \|f_{+,l}\|_{H^{-\alpha}} \|\widehat{E}^{\alpha} \widehat{S}^{\alpha} \widehat{t^{k+l} b_T g}\|_{L^2}, \end{aligned} \quad (4.15)$$

Through some computations, we have

$$\big\|\widehat{t^{k+l} b_T}\big\|_{L^1} \leq C(k+l) T^{k+l} \|b\|_{H^1}, \quad (4.16a)$$

$$\big\||\tau|^{2\alpha} \widehat{t^{k+l} b_T}\big\|_{L^1} \leq C(k+l) T^{k+l-2\alpha} \|b\|_{H^1}, \quad (4.16b)$$

provided that $\alpha < \dfrac{1}{4}$. With the aid of the above and the observation

$$\widehat{E}(\tau,\xi) \le |\tau-\lambda| + \widehat{E}(\lambda,\xi), \quad \widehat{S}(\tau,\xi) \le |\tau-\lambda| + \widehat{S}(\lambda,\xi), \tag{4.16c}$$

we can estimate

$$\begin{aligned}
&\|\widehat{E}^\alpha \widehat{S}^\alpha \widehat{t^{k+l} b_T g}\|_{L^2} \\
\le&\Big(\big\|\widehat{t^{k+l}b_T}\big\|_{L^1} + \big\||\tau|^{2\alpha}\widehat{t^{k+l}b_T}\big\|_{L^1}\Big)\|\widehat{E}^\alpha \widehat{S}^\alpha \widehat{g}\|_{L^2} \\
\le& C(k+l+1)T^{k+l-2\alpha}\|\widehat{E}^\alpha \widehat{S}^\alpha \widehat{g}\|_{L^2}.
\end{aligned} \tag{4.17}$$

This completes the proof. □

Lemma 4.2. *Let $\alpha < \dfrac{1}{4}$. The following estimate holds*

$$\begin{aligned}
&\Big\|\frac{\widehat{b}_T * \big(\delta_{\mp}^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_{\pm}^{(l)}\widehat{A}_{\pm,l}\big)}{\widehat{E}^\alpha \widehat{S}^\alpha}\Big\|_{L^2} \\
&\qquad\qquad \le C(k+l+1)T^{k+l-2\alpha}\|f_{\pm,k}\|_{H^{-\alpha}}\|f_{\pm,l}\|_{H^{-\alpha}}.
\end{aligned} \tag{4.18}$$

Proof. Using duality, we demonstrate the case $(+,+)$, while the case $(-,-)$ is being similar. We first compute the fractional term

$$\frac{\widehat{D}(|\xi|,\xi)\gamma^0\widehat{D}(|\eta|,\eta)}{|\xi||\eta|} = \begin{cases} 0, & \text{if } \xi\eta < 0, \\ 2\gamma^0 \mp 2\gamma^1, & \text{if } \xi\eta > 0, \end{cases} \tag{4.19a}$$

and observe that, for $\xi\eta > 0$,

$$\big|-|\xi|+|\eta|\big| + |\xi+\eta| + 1 \sim \max\{|\xi|,|\eta|\} + 1, \tag{4.19b}$$

$$\Big|\big|-|\xi|+|\eta|\big| - |\xi+\eta|\Big| + 1 \sim \min\{|\xi|,|\eta|\} + 1. \tag{4.19c}$$

Thus, in the same manner we have

$$\begin{aligned}
&\Big|\big\langle b_T\overline{Z}_{+,k}Z_{+,l}, g\big\rangle\Big| \\
=&\Big|\int \widehat{f}^{\dagger}_{+,k}(-\xi)\frac{\widehat{D}(|\xi|,-\xi)\gamma^0\widehat{D}(|\eta|,\eta)}{|\xi||\eta|}\widehat{f}_{+,l}(\eta)\overline{\widehat{t^{k+l}b_T}g}(-|\xi|+|\eta|,\xi+\eta)\,d\xi d\eta\Big| \\
\le& C\|f_{+,k}\|_{H^{-\alpha}}\|f_{+,l}\|_{H^{-\alpha}}\cdot \\
&\Big(\int(|\xi|+1)^{2\alpha}(|\eta|+1)^{2\alpha}\big|\widehat{t^{k+l}b_T}g(-|\xi|+|\eta|,\xi+\eta)\big|^2\,d\xi d\eta\Big)^{\frac{1}{2}} \\
\le& C\|f_{+,k}\|_{H^{-\alpha}}\|f_{+,l}\|_{H^{-\alpha}}\|\widehat{E}^\alpha\widehat{S}^\alpha\widehat{t^{k+l}b_T g}\|_{L^2} \\
\le& C(k+l+1)T^{k+l-2\alpha}\|f_{+,k}\|_{H^{-\alpha}}\|f_{+,l}\|_{H^{-\alpha}}\|\widehat{E}^\alpha\widehat{S}^\alpha\widehat{g}\|_{L^2},
\end{aligned} \tag{4.20}$$

□

Lemma 4.3. *Let $\delta > 0$. The following estimates hold*

$$\|f_{\pm,0}\|_{H^{-\alpha}} \leq C\Big(\|\psi_0\|_{H^{-\alpha}} + \Big\|\frac{\widehat{G}}{\widehat{M}^{\alpha}\widehat{S}^{\frac{1}{2}-\delta}}\Big\|_{L^2}\Big), \tag{4.21a}$$

$$\|f_{\pm,k}\|_{H^{-\alpha}} \leq C\frac{1}{k!}\|\frac{\widehat{G}_{\pm}}{\widehat{M}^{\alpha}\widehat{S}^{\frac{1}{2}-\delta}}\|_{L^2}. \tag{4.21b}$$

The proof for the Lemma 4.3 is straight forward so that we skip it. Notice that, in the (4.21b), $\widehat{S} \sim 1$ on the support of $\widehat{G}_{\pm}$.

Lemma 4.4. *With the notation above, the following estimate holds*

$$\Big\|\frac{\widehat{b}_T * \widehat{\overline{K}}_1 * \widehat{K}_1}{\widehat{E}^{\alpha}\widehat{S}^{\alpha}}\Big\|_{L^2} \leq C\Big\|\frac{\widehat{G}_f}{\widehat{M}^{\alpha}\widehat{S}^{\frac{1}{4}}}\Big\|^2_{L^2}. \tag{4.22}$$

Proof. For simplicity, we write $\widehat{G} := \widehat{G}_f$ and $\widehat{K} := \widehat{K}_1$. We use dyadic decomposition to handle this case. Assume that

$$\widehat{G} = \sum_{k=1}^{\infty} \widehat{G}_{\pm,\pm,k}, \tag{4.23}$$

where $\widehat{G}_{\pm,\pm,k}(\tau,\xi)$ is supported in one of the following types of regions:

$$\Sigma_{+,+} := \{(\tau,\xi) : \tau > 0, +2^{k-1} < \tau - |\xi| < +2^{k+1}\}, \tag{4.24a}$$

$$\Sigma_{+,-} := \{(\tau,\xi) : \tau > 0, -2^{k+1} < \tau - |\xi| < -2^{k-1}\}, \tag{4.24b}$$

$$\Sigma_{-,+} := \{(\tau,\xi) : \tau < 0, +2^{k-1} < \tau + |\xi| < +2^{k+1}\}, \tag{4.24c}$$

$$\Sigma_{-,-} := \{(\tau,\xi) : \tau < 0, -2^{k+1} < \tau + |\xi| < -2^{k-1}\}. \tag{4.24d}$$

The decomposition of $\widehat{G}$ induces a decomposition for $\widehat{K}$, namely

$$\widehat{K}_{\pm,\pm,k} = \frac{\widehat{D}}{\widehat{W}}\widehat{G}_{\pm,\pm,k}. \tag{4.25a}$$

To compute the convolution in (4.22),

$$\begin{aligned}&\widehat{\overline{K}}_{\pm,\pm,k} * \widehat{K}_{\pm,\pm,l}(-\tau,-\xi)\\ =&\int \widehat{\overline{K}}_{\pm,\pm,k}(-\tau-\sigma,-\xi-\eta)\widehat{K}_{\pm,\pm,l}(\sigma,\eta)\,d\sigma d\eta\\ =&\int \widehat{K}^{\dagger}_{\pm,\pm,k}(\tau+\sigma,\xi+\eta)\gamma^0\widehat{K}_{\pm,\pm,l}(\sigma,\eta)\,d\sigma d\eta,\end{aligned} \tag{4.25b}$$

we have 16 cases resulted from (4.24a-d) and (4.25b) as follows.

$$\{(\tau,\sigma,\xi,\eta)\colon \tau+\sigma>0, \sigma>0, \tau+\sigma-|\xi+\eta|\sim\pm 2^k, \sigma-|\eta|\sim\pm 2^l\} \tag{4.26a}$$

$$\{(\tau,\sigma,\xi,\eta)\colon \tau+\sigma<0, \sigma<0, \tau+\sigma+|\xi+\eta|\sim\pm 2^k, \sigma+|\eta|\sim\pm 2^l\} \tag{4.26b}$$

$$\{(\tau,\sigma,\xi,\eta)\colon \tau+\sigma<0, \sigma>0, \tau+\sigma+|\xi+\eta|\sim\pm 2^k, \sigma-|\eta|\sim\pm 2^l\} \tag{4.26c}$$

$$\{(\tau,\sigma,\xi,\eta)\colon \tau+\sigma>0, \sigma<0, \tau+\sigma-|\xi+\eta|\sim\pm 2^k, \sigma+|\eta|\sim\pm 2^l\} \tag{4.26d}$$

We label them as

$$\Sigma_{k,l}[(\pm,\pm);(\pm,\pm)], \tag{4.27}$$

and denote by $\Sigma_{k,l}$ without specifying which one precisely. We also use $\widehat{K}_k$ for abbreviation of $\widehat{K}_{\pm,\pm,k}$ and $\widehat{G}_k$ for $\widehat{G}_{\pm,\pm,k}$.

Let g be an arbitrary function. We first compute

$$\begin{aligned}&\left[\gamma^0(\tau+\sigma)-\gamma^1(\xi+\eta)\right]\gamma^0\left[\gamma^0\sigma+\gamma^1\eta\right]\\ =&\gamma^0\left[(\tau+\sigma)\sigma-(\xi+\eta)\eta\right]+\gamma^1\left[(\tau+\sigma)\eta-\sigma(\xi+\eta)\right].\end{aligned} \tag{4.28}$$

Thus, we have

$$\begin{aligned}&\Big|\Big\langle \widehat{\overline{K}}_k * \widehat{K}_l, \widehat{g}\Big\rangle\Big|\\ =&\Big|\int \widehat{G}^{\dagger}_k(\tau+\sigma,\xi+\eta)\frac{\gamma^0(\tau+\sigma)-\gamma^1(\xi+\eta)}{(\tau+\sigma)^2-(\xi+\eta)^2}\gamma^0\frac{\gamma^0\sigma+\gamma^1\eta}{\sigma^2-\eta^2}\widehat{G}_l(\sigma,\eta)\cdot\\ &\overline{\widehat{g}}(-\tau,-\xi)d\sigma d\eta d\tau d\xi\Big|\\ \le& C\|\frac{\widehat{G}_k}{\widehat{M}^{\alpha}}\|_{L^2}\|\frac{\widehat{G}_l}{\widehat{M}^{\alpha}}\|_{L^2}\Big(\int I_{k,l}(\tau,\xi)\big|\widehat{g}(-\tau,-\xi)\big|^2 d\tau d\xi\Big)^{\frac{1}{2}},\end{aligned} \tag{4.29a}$$

where $I_{k,l}(\tau,\xi)$ is given by

$$I_{k,l}(\tau,\xi) := \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)Q(\tau,\sigma,\xi,\eta)}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta, \tag{4.29b}$$

and Q is given by the expression

$$Q(\tau,\sigma,\xi,\eta) := \big[(\tau+\sigma)\sigma - (\xi+\eta)\eta\big]^2 + \big[(\tau+\sigma)\eta - \sigma(\xi+\eta)\big]^2, \tag{4.29c}$$

and $D_{k,l}(\tau,\xi)$ is a slice of $\Sigma_{k,l}$ for fixed (τ,ξ), i.e.

$$D_{k,l}(\tau,\xi) := \{(\sigma,\eta) : (\tau,\sigma,\xi,\eta) \in \Sigma_{k,l}\}. \tag{4.29d}$$

We distinguish the cases into two sets,

$$\Sigma_{k,l}[(\pm,\cdot);(\pm,\cdot)] \quad \text{and} \quad \Sigma_{k,l}[(\pm,\cdot);(\mp,\cdot)], \tag{4.30}$$

due to the fact that the computation for the 8 cases in each set is similar. For simplicity, we will assume $k \geq l$, while the other case is similar.

Cases H. *We have the following estimate*

$$\left\| \frac{\widehat{\overline{\widehat{K}}}_{+,\cdot,k} * \widehat{K}_{+,\cdot,l}}{\widehat{E}^\alpha \widehat{S}^\alpha} \right\|_{L^2} \leq \frac{C}{2^{\frac{l}{2}}} \frac{1}{2^{(\frac{1}{2}-\alpha)k}} \left\| \frac{\widehat{G}_{+,\cdot,k}}{\widehat{M}^\alpha} \right\|_{L^2} \left\| \frac{\widehat{G}_{+,\cdot,l}}{\widehat{M}^\alpha} \right\|_{L^2}, \tag{4.31a}$$

$$\left\| \frac{\widehat{\overline{\widehat{K}}}_{-,\cdot,k} * \widehat{K}_{-,\cdot,l}}{\widehat{E}^\alpha \widehat{S}^\alpha} \right\|_{L^2} \leq \frac{C}{2^{\frac{l}{2}}} \frac{1}{2^{(\frac{1}{2}-\alpha)k}} \left\| \frac{\widehat{G}_{-,\cdot,k}}{\widehat{M}^\alpha} \right\|_{L^2} \left\| \frac{\widehat{G}_{-,\cdot,l}}{\widehat{M}^\alpha} \right\|_{L^2}. \tag{4.31b}$$

In these cases, we have $(\tau+\sigma)\sigma > 0$. Throughout some algebraic manipulation, the expression Q can be written as

$$\begin{aligned} 2Q =& (\tau+\sigma-|\xi+\eta|)^2(\sigma+|\eta|)^2 + (\tau+\sigma+|\xi+\eta|)^2(\sigma-|\eta|)^2 + \\ & 8(\tau+\sigma)\sigma\big[|\xi+\eta||\eta| - (\xi+\eta)\eta\big]. \end{aligned} \tag{4.32}$$

Take the case of

$$\widehat{\overline{\widehat{K}}}_{+,+,k} * \widehat{K}_{+,+,l},$$

as an example and in which $D_{k,l} = \{(\eta,\sigma) : \tau+\sigma-|\xi+\eta| \sim 2^k, \sigma-|\eta| \sim 2^l, (\tau,\sigma,\xi,\eta) \in \Sigma_{k,l}[(+,+);(+,+)]\}$. In this case $\tau+\sigma>0$ and $\sigma>0$. In the $\eta\sigma$-plane, this is the region of the intersection of two forward cones. One has the thickness of 2^k and the translation of $(-\xi,-\tau)$, while the other has thickness of 2^l. It is mostly bounded, except for the extreme case which is when one cone moves along the other cone such that the intersection region is unbounded. Denote the set $\widetilde{D}_{kl}$ to be the projection of the set $D_{k,l}$ onto the η-axis. When the set $D_{k,l}$ is bounded, two facts, $|\xi| = |\xi+\eta| + |\eta|$ and $|\widetilde{D}_{kl}| \le C2^k$, are available and will be used in the following estimates.

For the first part, we have

$$\begin{aligned} I^1_{k,l}(\tau,\xi) :=& \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma-|\xi+\eta|)^2(\sigma+|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta \\ =& \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(\tau+\sigma+|\xi+\eta|)^2(\sigma-|\eta|)^2} d\sigma d\eta \\ \le& \frac{C}{2^l}\int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(2^k+|\xi+\eta|)^2} d\eta. \end{aligned} \tag{4.33a}$$

Consider the case when $|\xi+\eta| \ge |\eta|$, we get

$$\begin{aligned} I^1_{k,l}(\tau,\xi) \le& \frac{C}{2^l}\int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(2^k+|\xi+\eta|)^2} d\eta \\ \le& \frac{C}{2^l}\int_{\widetilde{D}_{k,l}} \frac{(1+|\eta|)^{2\alpha}}{(2^k+|\xi+\eta|)^2} d\eta \widehat{E}^{2\alpha}(\tau,\xi) \\ \le& \frac{C}{2^l}\frac{1}{2^{(1-2\alpha)k}}\widehat{E}^{2\alpha}\widehat{S}^{2\alpha}. \end{aligned} \tag{4.33b}$$

The extreme case is that when one of the cones moves along the other, say down right, this will not cause any trouble. Here, the region $D_{k,l}$ is unbounded. For the case $|\xi+\eta| \le |\eta|$, we get

$$\begin{aligned} I^1_{k,l}(\tau,\xi) \le& \frac{C}{2^l}\int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(2^k+|\xi+\eta|)^2} d\eta \\ \le& \frac{C}{2^l}\int_{\widetilde{D}_{k,l}} \frac{(1+|\xi+\eta|)^{2\alpha}}{(2^k+|\xi+\eta|)^2} d\eta \widehat{E}^{2\alpha}(\tau,\xi) \\ \le& \frac{C}{2^l}\frac{1}{2^{(1-2\alpha)k}}\widehat{E}^{2\alpha}\widehat{S}^{2\alpha}. \end{aligned} \tag{4.33c}$$

Again the extreme case when one of the cones moves along the other cone, say up right, will not cause trouble. Here, the region $D_{k,l}$ is unbounded.

For the second part, we obtain

$$\begin{aligned}
I^2_{k,l}(\tau,\xi) :=& \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma+|\xi+\eta|)^2(\sigma-|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta \\
=& \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(\tau+\sigma-|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma d\eta \\
\leq& \frac{C}{2^{2k-l}} \int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(2^l+|\eta|)^2} d\eta. \qquad (4.34a)
\end{aligned}$$

Consider the case when $|\xi+\eta| \geq |\eta|$, we get

$$\begin{aligned}
I^2_{k,l}(\tau,\xi) \leq& \frac{C}{2^{2k-l}} \int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(2^l+|\eta|)^2} d\eta \\
\leq& \frac{C}{2^{2k-l}} \int_{\widetilde{D}_{k,l}} \frac{(|\eta|+1)^{2\alpha}}{(2^l+|\eta|)^2} d\eta \widehat{E}^{2\alpha} \\
\leq& \frac{C}{2^k} \frac{1}{2^{(1-2\alpha)l}} \widehat{E}^{2\alpha}\widehat{S}^{2\alpha}. \qquad (4.34b)
\end{aligned}$$

For the case $|\xi+\eta| \leq |\eta|$, we get

$$\begin{aligned}
I^2_{k,l}(\tau,\xi) \leq& \frac{C}{2^{2k-l}} \int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(2^k+|\xi+\eta|)^2} d\eta \\
\leq& \frac{C}{2^l} \int_{\widetilde{D}_{k,l}} \frac{(1+|\xi+\eta|)^{2\alpha}}{(2^l+|\eta|)^2} d\eta \widehat{E}^{2\alpha}(\tau,\xi) \\
\leq& \frac{C}{2^k} \frac{1}{2^{(1-2\alpha)l}} \widehat{E}^{2\alpha}\widehat{S}^{2\alpha}. \qquad (4.34c)
\end{aligned}$$

For the third part, we get

$$\begin{aligned}
I^3_{k,l}(\tau,\xi) :=& \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma)\sigma\big[|\xi+\eta||\eta|-(\xi+\eta)\eta\big]}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta \\
\leq& \frac{C}{2^{2k+2l}} \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma)\sigma|\xi+\eta||\eta|}{(\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma d\eta \\
\leq& \frac{C}{2^{2k+2l}} \int_{D_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}(\tau+\sigma)\sigma|\xi+\eta||\eta|}{(\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma d\eta \\
\leq& \frac{C}{2^{2k+l}} \int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}|\xi+\eta||\eta|}{(2^k+|\xi+\eta|)(2^l+|\eta|)} d\eta. \qquad (4.35a)
\end{aligned}$$

Consider the case when $|\xi+\eta| \geq |\eta|$, we have

$$\begin{aligned}
I^3_{k,l}(\tau,\xi) \leq & \frac{C}{2^{2k+l}} \int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}|\xi+\eta||\eta|}{(2^k+|\xi+\eta|)(2^l+|\eta|)}\, d\eta \\
\leq & \frac{C}{2^{2k+l}} \int_{\widetilde{D}_{k,l}} \frac{(|\eta|+1)^{2\alpha}|\xi+\eta||\eta|}{(2^k+|\xi+\eta|)(2^l+|\eta|)}\, d\eta \widehat{E}^{2\alpha} \\
\leq & \frac{C}{2^{2k+l}} \int_{\widetilde{D}_{k,l}} (|\eta|+1)^{2\alpha}\, d\eta \widehat{E}^{2\alpha} \\
\leq & \frac{C}{2^l} \frac{1}{2^{(1-2\alpha)k}} \widehat{E}^{2\alpha} \widehat{S}^{2\alpha}. \qquad (4.35b)
\end{aligned}$$

The extreme case will not cause trouble since $\xi+\eta$ and η are of the same sign except on a bounded region, i.e. $\big[|\xi+\eta||\eta| - (\xi+\eta)\eta\big] = 0$ except on a bounded region. For the case $|\xi+\eta| \leq |\eta|$, we get

$$\begin{aligned}
I^3_{k,l}(\tau,\xi) \leq & \frac{C}{2^{2k+l}} \int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}|\xi+\eta||\eta|}{(2^k+|\xi+\eta|)(2^l+|\eta|)}\, d\eta \\
\leq & \frac{C}{2^{2k+l}} \int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}|\xi+\eta||\eta|}{(2^k+|\xi+\eta|)(2^l+|\eta|)}\, d\eta \widehat{E}^{2\alpha} \\
\leq & \frac{C}{2^{2k+l}} \int_{\widetilde{D}_{k,l}} (|\xi+\eta|+1)^{2\alpha}\, d\eta \widehat{E}^{2\alpha} \\
\leq & \frac{C}{2^l} \frac{1}{2^{(1-2\alpha)k}} \widehat{E}^{2\alpha} \widehat{S}^{2\alpha}. \qquad (4.35c)
\end{aligned}$$

The extreme case will not cause trouble since $\xi+\eta$ and η are of the same sign except on a bounded region, i.e. $\big[|\xi+\eta||\eta| - (\xi+\eta)\eta\big] = 0$ except on a bounded region.

Cases E. *We have the following estimate*

$$\Big\| \frac{\widehat{\overline{\widehat{K}}_{-,\cdot,k} * \widehat{K}_{+,\cdot,l}}}{\widehat{E}^\alpha \widehat{S}^\alpha} \Big\|_{L^2} \leq \frac{C}{2^{\frac{l}{2}}} \frac{1}{2^{(\frac{1}{2}-\alpha)k}} \Big\| \frac{\widehat{G}_{-,\cdot,k}}{\widehat{M}^\alpha} \Big\|_{L^2} \Big\| \frac{\widehat{G}_{+,\cdot,l}}{\widehat{M}^\alpha} \Big\|_{L^2}, \quad (4.36a)$$

$$\Big\| \frac{\widehat{\overline{\widehat{K}}_{+,\cdot,k} * \widehat{K}_{-,\cdot,l}}}{\widehat{E}^\alpha \widehat{S}^\alpha} \Big\|_{L^2} \leq \frac{C}{2^{\frac{l}{2}}} \frac{1}{2^{(\frac{1}{2}-\alpha)k}} \Big\| \frac{\widehat{G}_{+,\cdot,k}}{\widehat{M}^\alpha} \Big\|_{L^2} \Big\| \frac{\widehat{G}_{-,\cdot,l}}{\widehat{M}^\alpha} \Big\|_{L^2}. \quad (4.36b)$$

In these cases, we have $(\tau+\sigma)\sigma < 0$. Throughout some algebraic manipulation, the expression Q can be written as

$$\begin{aligned}
2Q = & (\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2 + (\tau+\sigma-|\xi+\eta|)^2(\sigma-|\eta|)^2 - \\
& 8(\tau+\sigma)\sigma\big[|\xi+\eta||\eta| + (\xi+\eta)\eta\big]. \qquad (4.37)
\end{aligned}$$

Take the case of

$$\widehat{\widehat{K}}_{-,+,k} * \widehat{K}_{+,+,l},$$

as an example and in which $D_{k,l} = \{(\eta,\sigma) : \tau+\sigma+|\xi+\eta| \sim 2^k, \sigma-|\eta| \sim 2^l, (\tau,\sigma,\xi,\eta) \in \Sigma_{k,l}[(-,+);(+,+)]\}$. In this case $\tau+\sigma<0$ and $\sigma>0$. In $\eta\sigma$-plane, this is the region of the intersection of a truncated backward cone with a forward cone. One has the thickness of 2^k and the translation of $(-\xi,-\tau)$, while the other has thickness of 2^l. It is bounded for all cases. We still have the extreme case which is when one cone moves along the other cone, though the region of intersection can be as large as possible, nevertheless it is bounded.

Again for the first part, we can estimate

$$\begin{aligned} I^1_{k,l}(\tau,\xi) :=& \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta \\ =& \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(\tau+\sigma-|\xi+\eta|)^2(\sigma-|\eta|)^2} d\sigma d\eta \\ \le& \frac{C}{2^{2l}} \int_{D_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(\tau+\sigma-|\xi+\eta|)^2} d\sigma d\eta. \end{aligned} \tag{4.38a}$$

To estimate the above integral, we separate the cases for $|\xi+\eta| \ge |\eta|$, $|\xi+\eta| \le |\eta|$, and the extreme case. Throughout some calculations, in each case, we have

$$I^1_{k,l}(\tau,\xi) \le \frac{C}{2^l}\frac{1}{2^{(1-2\alpha)k}}\widehat{E}^{2\alpha}\widehat{S}^{2\alpha}. \tag{4.38b}$$

For the second part, we derive

$$\begin{aligned} I^2_{k,l}(\tau,\xi) :=& \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma-|\xi+\eta|)^2(\sigma-|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma d\eta \\ =& \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma d\eta \\ \le& \frac{C}{2^{2k}} \int_{D_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(\sigma+|\eta|)^2} d\sigma d\eta \\ \le& \frac{C2^l}{2^{2k}} \int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}}{(2^l+|\eta|)^{2+2\alpha}} d\eta. \end{aligned} \tag{4.39a}$$

To estimate the above integral, we separate the cases for $|\xi+\eta| \geq |\eta|$, $|\xi+\eta| \leq |\eta|$, and the extreme case. Throughout some calculations, in each case, we have

$$I^2_{k,l}(\tau,\xi) \leq \frac{C}{2^l}\frac{1}{2^{(1-2\alpha)k}}\widehat{E}^{2\alpha}\widehat{S}^{2\alpha}. \tag{4.39b}$$

For the third part, we have

$$\begin{aligned} I^3_{k,l}(\tau,\xi) :=& \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)|\tau+\sigma|\sigma\big[|\xi+\eta||\eta|+(\xi+\eta)\eta\big]}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)}d\sigma d\eta \\ \leq& \frac{C}{2^{2k+2l}}\int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)|\tau+\sigma|\sigma|\xi+\eta||\eta|}{(\tau+\sigma-|\xi+\eta|)^2(\sigma+|\eta|)^2}d\sigma d\eta \\ \leq& \frac{C}{2^{2k+2l}}\int_{D_{k,l}} (|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}\ d\sigma d\eta. \end{aligned} \tag{4.40a}$$

To estimate the above integral, we separate the cases for $|\xi+\eta| \geq |\eta|$, $|\xi+\eta| \leq |\eta|$, and the extreme case. Notice that for the extreme case, we have $|\xi+\eta||\eta|+(\xi+\eta)\eta = 0$ except on a small part of the region of the intersection. Throughout some calculations, in each case, we have

$$I^3_{k,l}(\tau,\xi) \leq \frac{C}{2^l}\frac{1}{2^{(1-2\alpha)k}}\widehat{E}^{2\alpha}\widehat{S}^{2\alpha}. \tag{4.40b}$$

Now we return to the proof of (4.22). Combine (4.31), (4.36), we get

$$\begin{aligned} \Big|\big\langle \overline{K}_k K_l, g\big\rangle\Big| \leq& C\Big\|\frac{\widehat{G}_k}{\widehat{M}^\alpha}\Big\|_{L^2}\Big\|\frac{\widehat{G}_l}{\widehat{M}^\alpha}\Big\|_{L^2}\Big(\int I_{k,l}(\tau,\xi)\big|\widehat{g}(-\tau,-\xi)\big|^2 d\tau d\xi\Big)^{\frac{1}{2}} \\ \leq& \frac{C}{2^{\frac{l}{2}}}\frac{1}{2^{(\frac{1}{2}-\alpha)k}}\Big\|\frac{\widehat{G}_k}{\widehat{M}^\alpha}\Big\|_{L^2}\Big\|\frac{\widehat{G}_l}{\widehat{M}^\alpha}\Big\|_{L^2}\|\widehat{E}^\alpha\widehat{S}^\alpha\widehat{g}\|_{L^2} \\ \leq& \frac{C}{2^{(\frac{1}{4}-\alpha)k+\frac{l}{4}}}\Big\|\frac{\widehat{G}_k}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\Big\|\frac{\widehat{G}_l}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\|\widehat{E}^\alpha\widehat{S}^\alpha\widehat{g}\|_{L^2}. \end{aligned} \tag{4.41}$$

Finally, we have

$$\begin{aligned} &\Big\|\frac{\widehat{\overline{K}}*\widehat{K}}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2} \leq \sum_{k,l}\Big\|\frac{\widehat{\overline{K}}_k*\widehat{K}_l}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2} \\ \leq& \sum_{k,l}\frac{C}{2^{(\frac{1}{4}-\alpha)k+\frac{l}{4}}}\Big\|\frac{\widehat{G}_k}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\Big\|\frac{\widehat{G}_l}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2} \leq C\Big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{4}}}\Big\|^2_{L^2}. \end{aligned} \tag{4.42}$$

This completes the proof. □

The estimates for the remaining cases are given in the following Lemma.

Lemma 4.5. *For $j = 1, 2$ and $k = 0, 1, 2, \cdots$. The following estimates hold*

$$\Big\| \frac{\widehat{b}_T * \big(\delta_{\mp}^{(k)} \widehat{\overline{A}}_{\pm,k}\big) * \big(\widehat{K}_j\big)}{\widehat{E}^{\alpha} \widehat{S}^{\alpha}} \Big\|_{L^2} \leq C(k+1) T^{k-\frac{1}{2}} \|f_{\pm,k}\|_{H^{-\alpha}} \Big\| \frac{\widehat{G}}{\widehat{M}^{\alpha} \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2}, \tag{4.43a}$$

$$\Big\| \frac{\widehat{b}_T \widehat{\overline{K}}_j * \big(\delta_{\pm}^{(k)} \widehat{A}_{\pm,k}\big)}{\widehat{E}^{\alpha} \widehat{S}^{\alpha}} \Big\|_{L^2} \leq C(k+1) T^{k-\frac{1}{2}} \|f_{\pm,k}\|_{H^{-\alpha}} \Big\| \frac{\widehat{G}}{\widehat{M}^{\alpha} \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2}, \tag{4.43b}$$

$$\Big\| \frac{\widehat{b}_T * \widehat{\overline{K}}_1 * \widehat{K}_2}{\widehat{E}^{\alpha} \widehat{S}^{\alpha}} \Big\|_{L^2} \leq C \Big\| \frac{\widehat{G}}{\widehat{M}^{\alpha} \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2}^2, \tag{4.43c}$$

$$\Big\| \frac{\widehat{b}_T * \widehat{\overline{K}}_2 * \widehat{K}_j}{\widehat{E}^{\alpha} \widehat{S}^{\alpha}} \Big\|_{L^2} \leq C \Big\| \frac{\widehat{G}}{\widehat{M}^{\alpha} \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2}^2, \tag{4.43d}$$

The proof of Lemma 4.5 is a repetition of the arguments presented in Lemmas 4.1, 4.2, and 4.4, so that we omit it.

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