A Critical Case on the Dirac-Klein-Gordon Equations in one Space Dimension

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Abstract. We establish local and global existence results for a critical case of Dirac-Klein-Gordon equations in one space dimension, employing a null form estimate, a bilinear estimate and a fixed point argument.

0. Introduction and Main Results.

In the present work, we like to study the Cauchy problem for the Dirac-Klein-Gordon equations. The unknown quantities are a spinor field $\psi : \mathbb{R} \times \mathbb{R}^1 \mapsto \mathbb{C}^4$ and a scalar field $\phi : \mathbb{R} \times \mathbb{R}^1 \mapsto \mathbb{R}$. The evolution equations for the fields are given below,

$$\mathcal{D}\psi = \phi\psi; \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^1 \tag{0.1a}$$

$$\Box\phi = \overline{\psi}\psi; \tag{0.1b}$$

$$\psi(0,x) := \psi_0(x), \quad \phi(0,x) = \phi_0(x), \quad \phi_t(0,x) = \phi_1(x), \tag{0.1c}$$

where $\mathcal{D}$ is the Dirac operator, $\mathcal{D} := -i\gamma^\mu\partial_\mu$, $\mu = 0, 1$, and γ^μ are the Dirac matrices, the wave operator $\Box = -\partial_{tt} + \partial_{xx}$, and $\overline{\psi} = \psi^\dagger\gamma^0$, and $\dagger$ is the complex conjugate transpose.

Besides demonstrating the usefulness of a null form estimate, the purpose of this work is to show an existence result for a critical case of the system, i.e. $\psi_0 \in L^2$, $\phi_0 \in H^{\frac{1}{2}}$, and $\phi_1 \in H^{-\frac{1}{2}}$. We will take advantage of the null form structure depicted in the nonlinear term $\overline{\psi}\psi$ and a bilinear estimate, see [BG].

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For the DKG system, there are many conserved quantities which are not positive definite, such as the energy,

$$\int \mathrm{Im}(\psi^{\dagger}\gamma^0\gamma^j\partial_j\psi) + \phi\overline{\psi}\psi - \frac{1}{2}(\phi_t^2 + |\nabla\phi|^2)dx.$$

Therefore they are not applicable to derive a priori estimates. However the known positive conserved quantity is the law of conservation of charge,

$$\int |\psi(t)|^2 \, dx \ = \text{constant} \tag{0.2}$$

which leads to the global existence result, once the local existence result is established, see [Bo] and [F2-4].

In '73, Chadam showed that the Cauchy problem for the DKG equations has a global unique solution for $\psi_0 \in H^1$, $\phi_0 \in H^1$, $\phi_1 \in L^2$, see [C]. In '93, Zheng proved that there exists a global weak solution to the Cauchy problem of a modified DKG equations, based on the technique of compensated compactness, with $\psi_0 \in L^2$, $\phi_0 \in H^1$, $\phi_1 \in L^2$, see [Z]. In '00, Bournaveas derived a new proof of a global existence for the DKG equations, based on a null form estimate, if $\psi_0 \in L^2$, $\phi_0 \in H^1$, $\phi_1 \in L^2$, see [B]. In '04, Fang gave a direct proof for (0.1), based on a variant null form estimate, and obtain local solution for $\psi_0 \in H^{-\frac{1}{4}+\epsilon}$, $\phi_0 \in H^{\frac{1}{2}+\epsilon}$, $\phi_1 \in H^{-\frac{1}{2}+\epsilon}$, and global solution for $\psi_0 \in L^2$, $\phi_0 \in H^{\frac{1}{2}+\epsilon}$, $\phi_1 \in H^{-\frac{1}{2}+\epsilon}$, see [F3]. In '06, Bournaveas and Gibbeson obtained the result of global existence that lower the regularity of scalar field $\phi_0 \in H^r$, $\phi_1 \in H^{r-1}$, where $\frac{1}{4} \le r < \frac{1}{2}$, while the spinor field $\psi_0 \in L^2$. However the method they used does not apply to the critical case. On the other hand, from the scaling invariant, we expect that the regularity of the initial data is $\psi_0 \in H^{-1}$, $\phi_0 \in H^{-\frac{1}{2}}$, $\phi_1 \in H^{-\frac{3}{2}}$. The scaling group is $\phi_\lambda(x,t) = \lambda^{-1}\phi(\frac{x}{\lambda}, \frac{t}{\lambda})$ and $\psi_\lambda(x,t) = \lambda^{-\frac{3}{2}}\psi(\frac{x}{\lambda}, \frac{t}{\lambda})$. The current new result for the local existence is $\psi_0 \in H^{-\frac{1}{4}+\epsilon}$ and $(\phi_0, \phi_1) \in H^{\frac{1}{2}+\epsilon} \times H^{-\frac{1}{2}+\epsilon}$ while for the global existence is $\psi_0 \in L^2$ and $(\phi_0, \phi_1) \in H^r \times H^{r-1}$ with $r \ge \dfrac{1}{4}$.

The norm used in current paper and [F3,4] defines a version Bourgain space $X^{s,b}$, whose norm is given by

$$\|f\|_{X^{s,b}} = \|(|\xi|+1)^s(\big||\tau|-|\xi|\big|+1)^b \widehat{f}\|_{L^2}.$$

The case $\epsilon = 0$ is critical in the following sense. Assuming that the initial data (ϕ_0, ϕ_1) are in $H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$ does not imply that $\phi(t, \cdot)$ is bounded. In fact, it is a BMO function. One of the motivations due to the observation made by M. Grillakis for proving the existence of global solution with low regularity, which is that the initial data of (0.1): $\psi_0 \in L^2$, $\phi_0 \in H^{\frac{1}{2}}$, $\phi_1 \in H^{-\frac{1}{2}}$, is a natural space for the problem from the point of view of Hamiltonian and it is the right space for the existence of an invariant measure , see [B] and [Ku]. Also from the point of view of the energy, we can have the following bound:

$$\Big|\int \phi\overline{\psi}\psi(t)dx\Big| \leq \|\phi(t)\|_{H^{\frac{1}{2}}} \|\overline{\psi}\psi(t)\|_{H^{-\frac{1}{2}}} \leq \|\phi(t)\|_{H^{\frac{1}{2}}} \|\psi(t)\|_{L^2}^2.$$

The outline of this paper is as follows. First we write down some solutions representations via Fourier transform. Next we state some a priori estimates of solutions for Dirac equation and for wave equation. Then we show a local result for (0.1), employing the null form estimate, a bilinear estimate, and together with other estimates derived previously, and a fixed point argument. Finally we show the global existence by invoking the conservation law of charge.

The main result in this work is as follows.

Theorem 0.1. *(Global Existence) If the initial data of (0.1)* $\psi_0 \in L^2$, $\phi_0 \in H^{\frac{1}{2}}$, $\phi_1 \in H^{-\frac{1}{2}}$, *then there is a unique global solution for (0.1) with*

$$\psi \in C^0\big([0,\infty), L^2(R)\big); \quad \phi \in C^0\big([0,\infty), H^{\frac{1}{2}}(R)\big) \cap C^1\big([0,\infty), H^{-\frac{1}{2}}(R)\big).$$

1. Solution Representation.

In what follows, we denote by (t, x) the time-space variables and by (τ, ξ) the dual variables with respect to the Fourier transform of a given function. We will also often skip the constant in the inequalities. For convenience, we denote the multipliers by

$$\widehat{E}(\tau,\xi)=|\tau|+|\xi|+1,\quad \widehat{S}(\tau,\xi)=\Big||\tau|-|\xi|\Big|+1 \tag{1.1a}$$

$$\widehat{W}(\tau,\xi)=\tau^2-|\xi|^2,\qquad \widehat{D}(\tau,\xi)=\gamma^0\tau+\gamma^1\xi \tag{1.1b}$$

$$\widehat{M}(\xi)=|\xi|+1 \tag{1.1c}$$

Notice that $\widehat{W}$ and $\widehat{D}$ are the symbols of the wave and Dirac operators respectively.

Consider the Dirac equation,

$$\begin{cases} \mathcal{D}\psi=G, \quad (t,x)\in\mathbb{R}^1\times\mathbb{R}^1, \\ \psi(0)=\psi_0. \end{cases} \tag{1.2}$$

Let $\widehat{a}(\tau)$ be a cut-off function and equal 1 if $|\tau|\le\frac{1}{2}$ and equal 0 if $|\tau|\ge 1$, and denote by $h(\tau)$ the Heaviside function. For simplicity, let us write

$$\widehat{G}_\pm(\tau,\xi):=h(\pm\tau)\widehat{a}(\tau\mp|\xi|)\widehat{G}(\tau,\xi), \tag{1.3a}$$

$$\widehat{G}_f(\tau,\xi):=\widehat{G}(\tau,\xi)-\big(\widehat{G}_+(\tau,\xi)+\widehat{G}_-(\tau,\xi)\big), \tag{1.3b}$$

$$\widehat{D}_\pm:=\widehat{D}(|\xi|,\pm\xi). \tag{1.3c}$$

Notice that $\widehat{G}_\pm$ are supported in the regions $\{(\tau,\xi):\pm\tau>0, |\tau\mp|\xi||\le 1\}$ respectively and the decomposition of the forcing term is

$$\widehat{G}=\widehat{G}_f+\widehat{G}_++\widehat{G}_-.$$

Thus we can give a formula for $\widehat{\psi}$, namely

$$\widehat{\psi}(\tau,\xi)=\sum_{k=0}^{\infty}\Big(\delta_+^{(k)}(\tau,\xi)\widehat{A}_{+,k}(\xi)+\delta_-^{(k)}(\tau,\xi)\widehat{A}_{-,k}(\xi)\Big)+\widehat{K}(\tau,\xi), \tag{1.4}$$

where $\delta_\pm(\tau,\xi)$ are the delta functions supported on $\{\tau=\pm|\xi|\}$ respectively, $\delta^{(k)}$ mean derivatives of the delta function, and

$$\widehat{K}(\tau,\xi):=\frac{\widehat{D}(\tau,\xi)}{\widehat{W}(\tau,\xi)}\widehat{G}_f+\frac{(1-\widehat{a}_6(\tau))\widehat{D}_+\widehat{G}_-}{2|\xi|(\tau-|\xi|)}+\frac{(1-\widehat{a}_6)\widehat{D}_-\widehat{G}_+}{2|\xi|(\tau+|\xi|)}, \tag{1.5a}$$

$$\widehat{A}_{\pm,0}(\xi):=\frac{\widehat{D}_\pm}{2|\xi|}\Big[\gamma^0\widehat{\psi}_0-\int\frac{\widehat{G}_f+(1-\widehat{a}_6(\lambda))\widehat{G}_\mp}{\lambda\mp|\xi|}d\lambda\Big], \tag{1.5b}$$

$$\widehat{A}_{\pm,k}(\xi):=\frac{\widehat{D}_\pm(-1)^k}{2|\xi|k!}\int(\lambda\mp|\xi|)^{k-1}\big[\widehat{G}_\pm+\widehat{a}_6(\lambda)\widehat{G}_\mp\big]d\lambda, \tag{1.5c}$$

where $\widehat{a}_6(\tau) = \widehat{a}(\frac{\tau}{6})$

Consider the wave equation,

$$\begin{cases} \Box\phi = F, \quad (t,x) \in \mathbb{R}^1 \times \mathbb{R}^1, \\ \phi(0) = \phi_0, \;\; \phi_t(0) = \phi_1. \end{cases} \tag{1.6}$$

We can give an analogous formula for $\widehat{\phi}$, namely

$$\widehat{\phi}(\tau,\xi) = \sum_{k=0}^{\infty} \Big(\delta_+^{(k)}(\tau,\xi)\widehat{B}_{+,k}(\xi) + \delta_-^{(k)}(\tau,\xi)\widehat{B}_{-,k}(\xi) \Big) + \widehat{L}(\tau,\xi), \tag{1.7}$$

where $\delta_\pm(\tau,\xi)$ and $\delta^{(k)}$ are same as above, and

$$\widehat{L}(\tau,\xi) := \frac{\widehat{F}_f}{\widehat{W}(\tau,\xi)} - \frac{(1-\widehat{a}_6(\tau))\widehat{F}_-}{2|\xi|(|\xi|-\tau)} - \frac{(1-\widehat{a}_6(\tau))\widehat{F}_+}{2|\xi|(|\xi|+\tau)}, \tag{1.8a}$$

$$\widehat{B}_{\pm,0}(\xi) := \frac{1}{2|\xi|}\Big[\widehat{\phi}_\pm + \int \frac{\widehat{F}_f + (1-\widehat{a}_6(\lambda))\widehat{F}_\mp}{|\xi| \mp \lambda} d\lambda \Big], \tag{1.8b}$$

$$\widehat{B}_{\pm,k}(\xi) := \frac{\pm(-1)^k}{2|\xi|k!} \int (\lambda \mp |\xi|)^{k-1} \big[\widehat{F}_\pm + \widehat{a}_6(\lambda)\widehat{F}_\mp \big] d\lambda. \tag{1.8c}$$

2. Estimates.

To localize the solution in time, let $b(t)$ be a cut-off function such that $b(t)$ equals 1 if $|t| \le \frac{1}{2}$, and equals 0 if $|t| > 1$, and $b_T(t) = b(t/T)$. For an arbitrary function $f(t,x)$, we have

$$\|\widehat{b}_T * \widehat{f}\|_{L^2} = \|b_T f\|_{L^2} \le \|b_T\|_{L^\infty} \|f\|_{L^2}. \tag{2.1}$$

Lemma 2.1. *Let $\alpha = \frac{1}{4} - \epsilon$ and $\epsilon > 0$. If $\psi_0 \in H^{-\alpha}$, then we have*

$$\Big\| \widehat{b}_T * [\widehat{M}^{-\alpha}\widehat{S}^{\frac{3}{4}}\widehat{\psi}] \Big\|_{L^2(\mathbb{R}^1\times\mathbb{R}^1)} \le C\Big(\|\psi_0\|_{H^{-\alpha}} + \Big\| \frac{\widehat{G}}{\widehat{M}^\alpha \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2} \Big). \tag{2.2}$$

For the proof please see [F3].

Consider two Dirac equations,

$$\begin{cases} \mathcal{D}\psi_j = G_j, \quad j = 1,2, \\ \psi_j(0) = \psi_{0j}. \end{cases} \tag{2.3}$$

For the solutions of (2.3), we have the following key estimate whose proof can be found in [F3].

Lemma 2.2. *(Null Form Estimate) Let $\alpha = \frac{1}{4} - \epsilon$, $\epsilon > 0$, and ψ_1, ψ_2 be the solutions for (2.3). If $\psi_{0j} \in H^{-\alpha}$, we have*

$$\Big\|\frac{\widehat{(b_T\overline{\psi}_1\psi_2)}}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2} \leq C(T)\Big(\|\psi_{01}\|_{H^{-\alpha}} + \Big\|\frac{\widehat{G}_1}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\Big)\cdot \\ \Big(\|\psi_{02}\|_{H^{-\alpha}} + \Big\|\frac{\widehat{G}_2}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\Big). \quad (2.4)$$

For the wave equation (1.6), we have the following estimate.

Lemma 2.3. *Let ϕ be the solution of (1.6). If $\phi_0 \in H^{1-2\alpha}$ and $\phi_1 \in H^{-2\alpha}$, then*

$$\Big\|\widehat{b}_T * \Big[\widehat{M}^{-\alpha}(\widehat{E}\widehat{S})^{1-\alpha}\widehat{\phi}\Big]\Big\|_{L^2} \\ \leq C\Big(\|\phi_0\|_{H^{1-2\alpha}} + \|\phi_1\|_{H^{-2\alpha}} + \Big\|\frac{\widehat{F}}{\widehat{M}^\alpha\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2}\Big). \quad (2.5)$$

For the proof, please see [F3].

Lemma 2.4. *(Bournaveas & Gibbeson) Fix initial data $\psi_0 \in H^{-\frac{1}{4}}(R)$, $\phi_0 \in H^{\frac{1}{4}}(R)$, $\phi_1 \in H^{-\frac{3}{4}}(R)$, $G \in L^1([0,T]; H^{-\frac{1}{4}}(R))$, and $F \in L^1([0,T]; H^{-\frac{3}{4}}(R))$. Let the 2-spinor field ζ solve*

$$\mathcal{D}\zeta = i\phi\psi, \quad \zeta(0,\cdot) = 0, \quad (2.6)$$

where the scalar field ϕ and the 2-spinor field ψ solve

$$\mathcal{D}\psi = G, \quad \psi(0,\cdot) = \psi_0, \quad (2.7a)$$

$$\Box\phi = F, \quad \phi(0,\cdot) = \phi_0, \quad \phi_t(0,\cdot) = \phi_1. \quad (2.7b)$$

Then, for each $t \in [0,T]$, we have

$$\|\zeta(t)\|_{L^2(R)} \leq C(t)\Big[\|\psi_0\|_{H^{-\frac{1}{4}}(R)} + \int_0^t \|G(s,\cdot)\|_{H^{-\frac{1}{4}}(R)}\, ds\Big]\cdot \\ \Big[\|\phi_0\|_{H^{\frac{1}{4}}(R)} + \|\psi_1\|_{H^{-\frac{3}{4}}(R)} + \int_0^t \|F(s,\cdot)\|_{H^{-\frac{3}{4}}(R)}\, ds\Big]. \quad (2.8)$$

For the proof please see [BG]. We will also need some technical lemmata.

Lemma 2.5. *Let $f(t,x)$ and $g(t,x)$ be any functions such that $f \in L^q(L^2(\mathbb{R}))$ and $\widehat{S}^\beta \widehat{g} \in L^2(L^2(\mathbb{R}))$. Assume that $\delta \geq 0$, $q = \dfrac{8}{5-4\delta}$, $\dfrac{1}{r} = \dfrac{1}{2} - \beta$, and $2 \leq r \leq \infty$. Then we have*

$$\Big\| \frac{\widehat{b}_T * \widehat{f}}{\widehat{S}^{\frac14 - \delta}} \Big\|_{L^2} \leq C \|b_T f\|_{L^q(L^2)}, \tag{2.9}$$

$$\|g\|_{L^r(L^2)} \leq C \|\widehat{S}^\beta \widehat{g}\|_{L^2(L^2)}. \tag{2.10}$$

For the proof please see [F3].

3. Existence.

Now we are ready to prove the local existence for the (DKG) equations.

Theorem 3.1. *(Local Existence) If the initial data of (0.1) $\psi_0 \in L^2$, $\phi_0 \in H^{\frac12}$, $\phi_1 \in H^{-\frac12}$, then there is a unique local solution for (0.1).*

Proof. Consider the DKG problem

$$\mathcal{D}\psi = b_T \phi \psi; \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^1 \tag{3.1a}$$

$$\Box \phi = b_T \overline{\psi} \psi; \tag{3.1b}$$

$$\psi(0,x) := \psi_0(x), \quad \phi(0,x) = \phi_0(x), \quad \phi_t(0,x) = \phi_1(x), \tag{3.1c}$$

Iteration scheme induces a map $\mathcal{T}$ defined by

$$\mathcal{T}(\psi^k, \phi^k) = (\psi^{k+1}, \phi^{k+1}). \tag{3.2a}$$

We want to show that $\mathcal{T}$ is a contraction under the norm

$$\mathcal{N}(\psi,\phi) = \sup_{0 \leq t \leq T} \big\|\psi(t)\big\|_{L^2} + \big\| \widehat{E}^{\frac12} \widehat{S}^{\frac12 + \epsilon} \widehat{\phi} \big\|_{L^2}. \tag{3.2b}$$

For convenience, we call

$$J(0) = \|\phi_0\|_{H^{\frac12}} + \|\phi_1\|_{H^{-\frac12}} + \|\psi_0\|^2_{L^2} + 1. \tag{3.3}$$

First we apply (2.5) to compute

$$\Big\|\widehat{E}^{\frac12}\widehat{S}^{\frac12+\epsilon}\widehat{\phi}\Big\|_{L^2} \le C\Big(J(0)+\Big\|\frac{\widehat{b_T\overline{\psi}\psi}}{\widehat{M}^{\frac14+\epsilon}\widehat{E}^{\frac14-\epsilon}\widehat{S}^{\frac14-\epsilon}}\Big\|_{L^2}\Big). \tag{3.4}$$

Then we use (2.4) to get

$$\Big\|\frac{\widehat{b_T\overline{\psi}\psi}}{\widehat{M}^{\frac14+\epsilon}\widehat{E}^{\frac14-\epsilon}\widehat{S}^{\frac14-\epsilon}}\Big\|_{L^2} \le C\Big(J(0)+\Big\|\frac{\widehat{b_T\phi\psi}}{\widehat{M}^{\alpha}\widehat{S}^{\frac14}}\Big\|^2_{L^2}\Big). \tag{3.5}$$

By (2.8) and Sobolev inequality, we obtain

$$\Big\|\frac{\widehat{b_T\phi\psi}}{\widehat{M}^{\alpha}\widehat{S}^{\frac14}}\Big\|^2_{L^2} \le C\Big\|\frac{b_T\widetilde{\phi\psi}}{\widehat{M}^{\alpha}}\Big\|^2_{L^{\frac85}([0,T],L^2)} \le C\Big\|b_T\phi\psi\Big\|^2_{L^{\frac58}([0,T],L^{\frac{4}{3-4\epsilon}})}. \tag{3.6}$$

Thus we can bound

$$\Big\|b_T\phi\psi\Big\|^2_{L^{\frac58}([0,T],L^{\frac{4}{3-4\epsilon}})} \le C\|\phi\|^2_{L^{r_1}(L^{\frac{4}{1-4\epsilon}})}\|\psi\|^2_{L^{r_2}(L^2)}, \tag{3.7}$$

where $\dfrac{1}{r_1}+\dfrac{1}{r_2}=\dfrac58$. Finally, we have

$$\Big\|\widehat{E}^{\frac12}\widehat{S}^{\frac12+\epsilon}\widehat{\phi}\Big\|_{L^2} \le C\Big(J(0)+T^{\sigma}\big\|\widehat{E}^{\frac12}\widehat{S}^{\frac12+\epsilon}\widehat{\phi}\big\|^2_{L^2}\sup_{0\le t\le T}\big\|\psi(t)\big\|^2_{L^2}\Big), \tag{3.8}$$

where $\sigma>0$.

Next we want to bound the term involved with $\widehat{\psi}$. First we compute

$$\begin{aligned}\int_0^T \|\phi\psi(t,\cdot)\|_{H^{-\frac14}(R)}\,dt \le& \int_0^T \|\phi(t)\|_{H^{\frac14}}\|\psi(t)\|_{L^2}\,dt\\ \le& \|\phi\|_{L^1(H^{\frac14})}\sup\|\psi(t)\|_{L^2}\\ \le& T^{\sigma}\big\|\widehat{E}^{\frac12}\widehat{S}^{\frac12+\epsilon}\widehat{\phi}\big\|_{L^2}\sup_{0\le t\le T}\big\|\psi(t)\big\|_{L^2}\end{aligned} \tag{3.9}$$

and

$$\int_0^T \|\overline{\psi}\psi(t)\|_{H^{-\frac34}}\,dt \le \int_0^T \|\psi(t)\|^2_{L^2}\,dt \le T\sup\|\psi(t)\|^2_{L^2}. \tag{3.10}$$

The estimate given in (2.8) implies that

$$
\begin{aligned}
\|\psi(t)\|_{L^2(R)} &\leq \Big(\|\psi_0\|_{H^{-\frac{1}{4}}(R)} + \int_0^T \|\phi\psi(s,\cdot)\|_{H^{-\frac{1}{4}}(R)}\, ds\Big)\cdot \\
&\qquad \Big(\|\phi_0\|_{H^{\frac{1}{4}}(R)} + \|\psi_1\|_{H^{-\frac{3}{4}}(R)} + \int_0^T \|\overline{\psi}\psi(s,\cdot)\|_{H^{-\frac{3}{4}}(R)}\, ds\Big) \\
&\leq \Big(J(0) + T^\sigma \big\|\widehat{E}^{\frac{1}{2}}\widehat{S}^{\frac{1}{2}+\epsilon}\widehat{\phi}\big\|_{L^2} \sup_{0\leq t\leq T} \big\|\psi(t)\big\|_{L^2}\Big)\Big(J(0) + T\sup \|\psi(t)\|_{L^2}^2\Big) \\
&\leq C(T)\Big(J(0) + T^\sigma \mathcal{N}^2(\psi,\phi)\Big)\cdot \Big(J(0) + T\mathcal{N}^2(\psi,\phi)\Big). \qquad (3.11)
\end{aligned}
$$

Hence, using (3.8) and (3.11), we have

$$
\mathcal{N}(\mathcal{T}(\psi,\phi)) \leq C\big(J(0) + T^\sigma N^4(\psi,\phi)\big). \qquad (3.12)
$$

Choosing sufficiently large L, for suitable T, we have

$$
\mathcal{N}(\psi,\phi) \leq L \Longrightarrow N\big(\mathcal{T}(\psi,\phi)\big) \leq L, \qquad (3.13)
$$

provided that

$$
C(J(0) + T^\sigma L^4) \leq L. \qquad (3.14)
$$

Now we consider the difference $\mathcal{T}(\psi,\phi) - \mathcal{T}(\psi',\phi')$. Base on the observations

$$
\begin{aligned}
\overline{\psi}\psi - \overline{\psi'}\psi' &= \frac{1}{2}(\overline{\psi-\psi'})(\psi+\psi') + \frac{1}{2}(\overline{\psi+\psi'})(\psi-\psi'), \qquad (3.15a) \\
\phi\psi - \phi'\psi' &= \frac{1}{2}(\phi-\phi')(\psi+\psi') + \frac{1}{2}(\phi+\phi')(\psi-\psi'), \qquad (3.15b)
\end{aligned}
$$

Analogously, we get

$$
\mathcal{N}\big(\mathcal{T}(\psi-\psi',\phi-\phi')\big) \leq CT^\sigma L^3 N(\psi-\psi',\phi-\phi'). \qquad (3.16)
$$

Therefore for suitable T, we obtain

$$
\mathcal{N}\big(\mathcal{T}(\psi-\psi',\phi-\phi')\big) \leq \frac{1}{2} N(\psi-\psi',\phi-\phi'), \qquad (3.18)
$$

provided that

$$CT^{\sigma}L^{3}\le\frac{1}{2}. \tag{3.18}$$

We can conclude that the map $\mathcal{T}$ is indeed a contraction with respect to the norm $\mathcal{N}$, thus it has a unique fixed point. □

We now prove the global existence.

Proof of Theorem 0.1. From the law of conservation of charge, we have

$$\sup_{[0,T]}\|\psi(t)\|_{L^2}=\|\psi_0\|_{L^2}. \tag{3.19}$$

To bound ϕ, we apply Sobolev inequality and the energy estimate to compute

$$\|\phi(t)\|_{L^{\frac{4}{1-4\epsilon}}}\le\|\phi(t)\|_{H^{\frac{1+4\epsilon}{4}}}\le J(0)+\int_0^T\|\overline{\psi}\psi(t)\|_{H^{\frac{-3+4\epsilon}{4}}}\,dt.$$

Then we can get

$$\int_0^T\|\overline{\psi}\psi(t)\|_{H^{\frac{-3+4\epsilon}{4}}}\,dt\le\int_0^T\|\overline{\psi}\psi(t)\|_{L^1}\,dt\le T\|\psi_0\|_{L^2}^2.$$

Now we can estimate ϕ as follows. First we employ (2.10) and (2.5) to derive

$$\begin{aligned}\|\phi(t)\|_{H^{\frac12}}&\le\|\phi\|_{L^\infty(H^{\frac12})}\le\left\|\widehat{E}^{\frac12}\widehat{S}^{\frac12+\epsilon}\widehat{\phi}\right\|_{L^2}\\&\le J(0)+\left\|\frac{\widehat{\overline{\psi}\psi}}{\widehat{M}^{\frac14+\epsilon}\widehat{E}^{\frac14-\epsilon}\widehat{S}^{\frac14-\epsilon}}\right\|_{L^2}\le J(0)+\left\|\frac{\widehat{\phi\psi}}{\widehat{M}^{\frac14-\epsilon}\widehat{S}^{\frac14}}\right\|_{L^2}^2.\end{aligned} \tag{3.22}$$

Invoke (2.9) and Sobolev inequality, we obtain

$$\begin{aligned}\left\|\frac{\widehat{\phi\psi}}{\widehat{M}^{\frac14-\epsilon}\widehat{S}^{\frac14}}\right\|_{L^2}^2&\le\left\|\frac{\widetilde{\phi\psi}}{\widehat{M}^{\frac14-\epsilon}}\right\|_{L^{\frac85}(L^2)}^2\le\left\|\phi\psi\right\|_{L^{\frac85}(L^{\frac{4}{3-4\epsilon}})}^2\\&\le\left\|\phi\right\|_{L^{r_1}(L^{\frac{4}{1-4\epsilon}})}^2\left\|\psi\right\|_{L^{r_2}(L^2)}^2.\end{aligned} \tag{3.23}$$

Finally we get

$$\|\phi(t)\|_{H^{\frac12}}\le J(0)+\Big(J(0)+T\|\psi_0\|_{L^2}^2\Big)^2\Big(T^{\rho}\|\psi_0\|_{L^2}\Big)^2, \tag{3.24}$$

where ρ is some positive number. The calculation for $\|\phi_t(t)\|_{H^{-\frac12}}$ is analogous. Thus the above bounds ensure us to proceed the construction of solution beyond T. □

4. Null Form Estimate.

In this section, we sketch the proof of the key estimate. For more details, please see [F3].

Lemma 2.2. *(Null Form Estimate) Let $\alpha = \frac{1}{4} - \epsilon$, $\epsilon > 0$, and ψ_1, ψ_2 be the solutions for the Dirac equations (2.3). If the initial data $\psi_{0j} \in H^{-\alpha}$, $j = 1, 2$, then we have*

$$\Big\| \frac{\widehat{b_T \overline{\psi}_1 \psi_2}}{\widehat{E}^\alpha \widehat{S}^\alpha} \Big\|_{L^2} \leq C(T) \Big(\|\psi_{01}\|_{H^{-\alpha}} + \Big\| \frac{\widehat{G}_1}{\widehat{M}^\alpha \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2} \Big) \cdot \Big(\|\psi_{02}\|_{H^{-\alpha}} + \Big\| \frac{\widehat{G}_2}{\widehat{M}^\alpha \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2} \Big). \tag{4.1}$$

The proof for the estimate is based on the duality argument and it will be given in a number of steps. Without loss of generality, we assume that $\psi_1 = \psi_2$, and prove: if ψ is a solution of the Dirac equation (1.2), then

$$\Big\| \frac{\widehat{b_T \overline{\psi} \psi}}{\widehat{E}^\alpha \widehat{S}^\alpha} \Big\|_{L^2} \leq C(T) \Big(\|\psi_0\|_{H^{-\alpha}} + \Big\| \frac{\widehat{G}}{\widehat{M}^\alpha \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2} \Big)^2. \tag{4.2}$$

The formula for $\widehat{\psi}$, as in (1.4), for the Dirac equation (1.2) is given by

$$\widehat{\psi}(\tau, \xi) = \sum_{k=0}^{\infty} \Big(\delta_+^{(k)}(\tau, \xi) \widehat{A}_{+,k}(\xi) + \delta_-^{(k)}(\tau, \xi) \widehat{A}_{-,k}(\xi) \Big) + \widehat{K}(\tau, \xi), \tag{4.3}$$

where $\delta_\pm(\tau, \xi)$ are the delta functions supported on $\{\tau = \pm|\xi|\}$ respectively, $\delta^{(k)}$ mean derivatives of the delta function, and

$$\widehat{K}(\tau, \xi) := \frac{\widehat{D}(\tau, \xi)}{\widehat{W}(\tau, \xi)} \widehat{G}_f + \frac{(1 - \widehat{a}_6(\tau)) \widehat{D}_+ \widehat{G}_-}{2|\xi|(\tau - |\xi|)} + \frac{(1 - \widehat{a}_6) \widehat{D}_- \widehat{G}_+}{2|\xi|(\tau + |\xi|)}, \tag{4.4a}$$

$$\widehat{A}_{\pm,0}(\xi) := \frac{\widehat{D}_\pm}{2|\xi|} \Big[\gamma^0 \widehat{\psi}_0 - \int \frac{\widehat{G}_f + (1 - \widehat{a}_6(\lambda)) \widehat{G}_\mp}{\lambda \mp |\xi|} d\lambda \Big], \tag{4.4b}$$

$$\widehat{A}_{\pm,k}(\xi) := \frac{\widehat{D}_\pm (-1)^k}{2|\xi| k!} \int (\lambda \mp |\xi|)^{k-1} \big[\widehat{G}_\pm + \widehat{a}_6(\lambda) \widehat{G}_\mp \big] d\lambda. \tag{4.4c}$$

Moreover we write

$$\widehat{A}_{\pm,k}(\xi) := \frac{\widehat{D}_\pm}{2|\xi|}\widehat{f}_{\pm,k}(\xi), \tag{4.5}$$

and split $\widehat{K} = \widehat{K}_1 + \widehat{K}_2$, where

$$\widehat{K}_1 := \frac{\widehat{D}(\tau,\xi)}{\widehat{W}(\tau,\xi)}\widehat{G}_f; \quad \widehat{K}_2 := \frac{b_1\widehat{D}_+\widehat{G}_- + b_2\widehat{D}_-\widehat{G}_+}{\widehat{E}\widehat{S}}, \tag{4.6}$$

and b_1, b_2 are bounded functions. The Fourier transform of the quadratic expression, $\widehat{\overline{\psi}\psi} = \widehat{\overline{\psi}} * \widehat{\psi}$, can be written as the sum of the following terms.

$$\sum_{k,l} \big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_\pm^{(l)}\widehat{A}_{\pm,l}\big), \tag{4.7a}$$

$$\sum_{k,l} \big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_\mp^{(l)}\widehat{A}_{\mp,l}\big), \tag{4.7b}$$

$$\sum_{k} \big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\widehat{K}_1 + \widehat{K}_2\big) + \big(\widehat{\overline{K}}_1 + \widehat{\overline{K}}_2\big) * \sum_k \big(\delta_\pm^{(k)}\widehat{A}_{\pm,k}\big), \tag{4.7c}$$

$$\widehat{\overline{K}}_1 * \widehat{K}_1 + \widehat{\overline{K}}_1 * \widehat{K}_2 + \widehat{\overline{K}}_2 * \widehat{K}_1 + \widehat{\overline{K}}_2 * \widehat{K}_2. \tag{4.7d}$$

Notice that

$$\widehat{A^\dagger_{\pm,k}}(\xi) = \widehat{A}^\dagger_{\pm,k}(-\xi); \qquad \widehat{f^+_{\pm,k}}(\xi) = \widehat{f}^\dagger_{\pm,k}(-\xi), \tag{4.8a}$$

$$\widehat{\overline{A}_{\pm,k}}(\xi) = \widehat{f}^\dagger_{\pm,k}(-\xi)\frac{\widehat{D}_\pm}{|\xi|}\gamma^0; \quad \widehat{\overline{K}}(\tau,\xi) = \widehat{K}^\dagger(-\tau,-\xi)\gamma^0, \tag{4.8b}$$

and

$$\widehat{\overline{\psi}}(\tau,\xi) = \sum_{k=0}^{\infty}\Big(\delta_-^{(k)}(\tau,\xi)\widehat{\overline{A}}_{+,k}(\xi) + \delta_+^{(k)}(\tau,\xi)\widehat{\overline{A}}_{-,k}(\xi)\Big) + \widehat{\overline{K}}(\tau,\xi), \tag{4.9}$$

Lemma 4.1. *Let $\alpha < \dfrac{1}{4}$. The following estimate holds*

$$\Big\| \frac{\widehat{b}_T * \big(\delta_\mp^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_\mp^{(l)}\widehat{A}_{\mp,l}\big)}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2} \le C(k+l+1)T^{k+l-2\alpha}\|f_{\pm,k}\|_{H^{-\alpha}}\|f_{\mp,l}\|_{H^{-\alpha}}. \tag{4.10}$$

Lemma 4.2. *Let* $\alpha < \dfrac{1}{4}$. *The following estimate holds*

$$\Big\| \frac{\widehat{b}_T * \big(\delta_{\mp}^{(k)} \widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_{\pm}^{(l)} \widehat{A}_{\pm,l}\big)}{\widehat{E}^{\alpha} \widehat{S}^{\alpha}} \Big\|_{L^2} \leq C(k+l+1) T^{k+l-2\alpha} \|f_{\pm,k}\|_{H^{-\alpha}} \|f_{\pm,l}\|_{H^{-\alpha}}. \tag{4.11}$$

Lemma 4.3. *Let* $\delta > 0$. *The following estimates hold*

$$\|f_{\pm,0}\|_{H^{-\alpha}} \leq C\Big(\|\psi_0\|_{H^{-\alpha}} + \Big\| \frac{\widehat{G}}{\widehat{M}^{\alpha} \widehat{S}^{\frac{1}{2}-\delta}} \Big\|_{L^2} \Big), \tag{4.12a}$$

$$\|f_{\pm,k}\|_{H^{-\alpha}} \leq C \frac{1}{k!} \| \frac{\widehat{G}_{\pm}}{\widehat{M}^{\alpha} \widehat{S}^{\frac{1}{2}-\delta}} \|_{L^2}. \tag{4.12b}$$

The proof for the Lemma 4.3 is straight forward so that we skip it. Notice that, in the (4.12b), $\widehat{S} \sim 1$ on the support of $\widehat{G}_{\pm}$.

Lemma 4.4. *With the notation above, the following estimate holds*

$$\Big\| \frac{\widehat{b}_T * \widehat{\overline{K}}_1 * \widehat{K}_1}{\widehat{E}^{\alpha} \widehat{S}^{\alpha}} \Big\|_{L^2} \leq C \Big\| \frac{\widehat{G}_f}{\widehat{M}^{\alpha} \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2}^2. \tag{4.13}$$

The estimates for the remaining cases are given in the following Lemma.

Lemma 4.5. *For* $j = 1, 2$ *and* $k = 0, 1, 2, \cdots$. *The following estimates hold*

$$\Big\| \frac{\widehat{b}_T * \big(\delta_{\mp}^{(k)} \widehat{\overline{A}}_{\pm,k}\big) * \big(\widehat{K}_j\big)}{\widehat{E}^{\alpha} \widehat{S}^{\alpha}} \Big\|_{L^2} \leq C(k+1) T^{k-\frac{1}{2}} \|f_{\pm,k}\|_{H^{-\alpha}} \Big\| \frac{\widehat{G}}{\widehat{M}^{\alpha} \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2}, \tag{4.14a}$$

$$\Big\| \frac{\widehat{b}_T \widehat{\overline{K}}_j * \big(\delta_{\pm}^{(k)} \widehat{A}_{\pm,k}\big)}{\widehat{E}^{\alpha} \widehat{S}^{\alpha}} \Big\|_{L^2} \leq C(k+1) T^{k-\frac{1}{2}} \|f_{\pm,k}\|_{H^{-\alpha}} \Big\| \frac{\widehat{G}}{\widehat{M}^{\alpha} \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2}, \tag{4.14b}$$

$$\Big\| \frac{\widehat{b}_T * \widehat{\overline{K}}_1 * \widehat{K}_2}{\widehat{E}^{\alpha} \widehat{S}^{\alpha}} \Big\|_{L^2} \leq C \Big\| \frac{\widehat{G}}{\widehat{M}^{\alpha} \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2}^2, \tag{4.14c}$$

$$\Big\| \frac{\widehat{b}_T * \widehat{\overline{K}}_2 * \widehat{K}_j}{\widehat{E}^{\alpha} \widehat{S}^{\alpha}} \Big\|_{L^2} \leq C \Big\| \frac{\widehat{G}}{\widehat{M}^{\alpha} \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2}^2, \tag{4.14d}$$

The proof of Lemma 4.5 is a repetition of the arguments in Lemmas 4.1, 4.2, and 4.4.

5. Pseudoscalar $\overline{\psi}\gamma^5\psi$.

Consider the Dirac-Klein-Gordon equations with the pseudoscalar nonlinear term:

$$\mathcal{D}\psi = \phi\psi; \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^1 \tag{5.1a}$$

$$\Box\phi = \overline{\psi}\gamma^5\psi; \tag{5.1b}$$

$$\psi(0,x) := \psi_0(x), \quad \phi(0,x) = \phi_0(x), \quad \phi_t(0,x) = \phi_1(x), \tag{5.1c}$$

where the matrix $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$. From the point of view of the null form estimate, Lemma 2.2, it is essentially same as the scalar term $\overline{\psi}\psi$ for the space dimension $n = 1, 2, 3$. Since

$$\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5, \quad \mu = 0,1,2,3, \tag{5.2a}$$

$$\widehat{D}_-(\xi)\gamma^0\gamma^5\widehat{D}_+(\eta) = \gamma^5\widehat{D}_-(\xi)\gamma^0\widehat{D}_+(\eta), \tag{5.2b}$$

$$\widehat{D}(\tau+\sigma,\xi+\eta)\gamma^0\gamma^5\widehat{D}(\tau,\eta) = \gamma^5\widehat{D}(\tau+\sigma,\xi+\eta)\gamma^0\widehat{D}(\tau,\eta). \tag{5.2c}$$

Lemma 5.1. *(Null Form Estimate) Let* $\alpha = \dfrac{1}{4} - \epsilon$, $\epsilon > 0$, *and* ψ_1, ψ_2 *be the solutions for (2.9). If* $\psi_{0j} \in H^{-\alpha}$, *we have*

$$\left\|\frac{\widehat{(b_T\overline{\psi}_1\gamma^5\psi_2)}}{\widehat{E}^\alpha\widehat{S}^\alpha}\right\|_{L^2} \le C(T)\Big(\|\psi_{01}\|_{H^{-\alpha}} + \Big\|\frac{\widehat{G}_1}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2}\Big)\cdot \Big(\|\psi_{02}\|_{H^{-\alpha}} + \Big\|\frac{\widehat{G}_2}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2}\Big). \tag{5.3}$$

Use the above method can prove the same result for the pseudoscalar case.

Theorem 5.1. *If the initial data* $\psi_0 \in H^{-\frac14+\epsilon}$, $\phi_0 \in H^{\frac12+\epsilon}$, $\phi_1 \in H^{-\frac12+\epsilon}$, *then we have a unique local solution for (5.1). If the initial data* $\psi_0 \in L^2$, $\phi_0 \in H^r$, $\phi_1 \in H^{r-1}$, *where* $\frac14 \le r$, *then we have a unique global solution for (5.1).*

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